Chapter 1

Section 1.1

1. This is a variables separable equation and $y = 0$ is a solution. If $y \neq 0$, then

$$\begin{aligned} \frac{dy}{y} &= \frac{2x}{x^2+1}\,dx \\ \Rightarrow\quad \ln|y| &= \ln(x^2+1) + C_0 \\ \Rightarrow\quad y(x) &= C(x^2+1), \end{aligned}$$

where $C \neq 0$. The value $C = 0$ generates the singular solution $y = 0$.

2. Proceeding as in **1**, for $y \neq -1$ we have

$$\begin{aligned} y' &= 3x^2(y+1) \\ \Rightarrow\quad \frac{dy}{y+1} &= 3x^2 \\ \Rightarrow\quad \ln|y+1| &= x^3 + C_0 \\ \Rightarrow\quad y &= Ce^{x^3} - 1, \end{aligned}$$

where $C \neq 0$. The value $C = 0$ generates the singular solution $y = -1$.

3. This is a linear equation, solved by the integrating factor method:

$$\begin{aligned} &\frac{dy}{dx} + \frac{2}{x-1}\,y = \frac{x}{x-1} \\ \Rightarrow\quad &\mu = \exp\left\{\int \frac{2}{x-1}\,dx\right\} = e^{2\ln|x-1|} = (x-1)^2 \\ \Rightarrow\quad &y(x) = \frac{1}{(x-1)^2}\int (x-1)^2 \frac{x}{x-1}\,dx = \frac{1}{(x-1)^2}\int (x^2 - x)\,dx \\ &\qquad = (x-1)^{-2}\left(\tfrac{1}{3}\,x^3 - \tfrac{1}{2}\,x^2 + C\right). \end{aligned}$$

4. Proceeding as in **3**, we have

$$\begin{aligned} &\frac{dy}{dx} - \frac{2}{x}\,y = x^3 e^x \\ \Rightarrow\quad &\mu = \exp\left\{\int -\frac{2}{x}\,dx\right\} = e^{-\ln x^2} = \frac{1}{x^2} \\ \Rightarrow\quad &y(x) = x^2 \int \frac{1}{x^2}\,x^3 e^x\,dx = x^2 \int xe^x\,dx \\ &\qquad = x^2\left\{xe^x - \int e^x\,dx\right\} = x^2(xe^x - e^x + C). \end{aligned}$$

Section 1.2

1. This is a homogeneous first-order linear equation with constant coefficients, solved by means of the corresponding characteristic equation:

$$2s+5=0 \quad\Rightarrow\quad s=-\tfrac{5}{2}$$
$$\Rightarrow\quad y(x)=C\,e^{-5x/2}.$$

2. Proceeding as in **1**, we have

$$3s-2=0 \quad\Rightarrow\quad s=\tfrac{2}{3}$$
$$\Rightarrow\quad y(x)=Ce^{2x/3}.$$

3. This is a homogeneous second-order linear equation with constant coefficients, so

$$s^2-4s+3=0 \quad\Rightarrow\quad s_1=1,\;\; s_2=3$$
$$\Rightarrow\quad y(x)=C_1e^{x}+C_2e^{3x}.$$

4. Proceeding as in **3**, we have

$$2s^2-5s+2=0 \quad\Rightarrow\quad s_1=2,\;\; s_2=\tfrac{1}{2}$$
$$\Rightarrow\quad y(x)=C_1e^{2x}+C_2e^{x/2}.$$

5. Proceeding as in **3**, we have

$$4s^2+4s+1=0 \quad\Rightarrow\quad s_1=s_2=-\tfrac{1}{2}$$
$$\Rightarrow\quad y(x)=(C_1+C_2x)e^{-x/2}.$$

6. Proceeding as in **5**, we have

$$s^2-6s+9=0 \quad\Rightarrow\quad s_1=s_2=3$$
$$\Rightarrow\quad y(x)=(C_1+C_2x)e^{3x}.$$

7. Proceeding as in **3**, we have

$$s^2+2s+5=0 \quad\Rightarrow\quad s_{1,2}=-1\pm 2i$$
$$\Rightarrow\quad y(x)=e^{-x}\big[C_1\cos(2x)+C_2\sin(2x)\big].$$

8. Proceeding as in **7**, we have

$$s^2-6s+13=0 \quad\Rightarrow\quad s_{1,2}=3\pm 2i$$
$$\Rightarrow\quad y(x)=e^{3x}\big[C_1\cos(2x)+C_2\sin(2x)\big].$$

Section 1.3

1. This is a nonhomogeneous linear equation with constant coefficients, so we use the ‘complementary function + particular integral’ method:

$$s+2=0 \quad\Rightarrow\quad s=-2$$
$$\Rightarrow\quad y_{CF}=Ce^{-2x},$$

$$
\begin{aligned}
& y_{PI} = ax + b + ce^{4x} \\
\Rightarrow\quad & (a + 4ce^{4x}) + 2(ax + b + ce^{4x}) = 2x + e^{4x} \\
\Rightarrow\quad & 2ax + (a + 2b) + 6ce^{4x} = 2x + e^{4x} \\
\Rightarrow\quad & 2a = 2, \quad a + 2b = 0, \quad 6c = 1 \\
\Rightarrow\quad & a = 1, \quad b = -\tfrac{1}{2}, \quad c = \tfrac{1}{6} \\
\Rightarrow\quad & y_{PI} = x - \tfrac{1}{2} + \tfrac{1}{6}\, e^{4x} \\
\Rightarrow\quad & y(x) = Ce^{-2x} + x - \tfrac{1}{2} + \tfrac{1}{6}\, e^{4x}.
\end{aligned}
$$

2. Proceeding as in **1**, we have

$$
\begin{aligned}
& 2s - 3 = 0 \quad \Rightarrow \quad s = \tfrac{3}{2} \\
\Rightarrow\quad & y_{CF} = Ce^{3x/2}, \\
& y_{PI} = ax + b + ce^{x} = x + 2 - e^{x} \\
\Rightarrow\quad & y(x) = Ce^{3x/2} + x + 2 - e^{x}.
\end{aligned}
$$

3. Proceeding as in **1**, we have

$$
\begin{aligned}
& 2s - 1 = 0 \quad \Rightarrow \quad s = \tfrac{1}{2} \\
\Rightarrow\quad & y_{CF} = Ce^{x/2}, \\
& y_{PI} = axe^{x/2} \\
\Rightarrow\quad & 2a\Big(e^{x/2} + \tfrac{1}{2}\, xe^{x/2}\Big) - axe^{x/2} = 2ae^{x/2} = e^{x/2} \\
\Rightarrow\quad & 2a = 1 \\
\Rightarrow\quad & a = \tfrac{1}{2} \\
\Rightarrow\quad & y_{PI} = \tfrac{1}{2}\, xe^{x/2} \\
\Rightarrow\quad & y(x) = Ce^{x/2} + \tfrac{1}{2}\, xe^{x/2} = \Big(C + \tfrac{1}{2}\, x\Big)e^{x/2}.
\end{aligned}
$$

4. Proceeding as in **3**, we have

$$
\begin{aligned}
& s + 1 = 0 \quad \Rightarrow \quad s = -1 \\
\Rightarrow\quad & y_{CF} = Ce^{-x}, \\
& y_{PI} = ax + b + cxe^{-x} = 1 - x + 2xe^{-x} \\
\Rightarrow\quad & y(x) = Ce^{-x} + 1 - x + 2xe^{-x}.
\end{aligned}
$$

5. Proceeding as in **1**, we have

$$
\begin{aligned}
& s^2 - 1 = 0 \quad \Rightarrow \quad s_{1,2} = \pm 1 \\
\Rightarrow\quad & y_{CF} = C_1 \cosh x + C_2 \sinh x, \\
& y_{PI} = ax^2 + bx + c \\
\Rightarrow\quad & 2a - (ax^2 + bx + c) = -ax^2 - bx + (2a - c) = x^2 - x + 2 \\
\Rightarrow\quad & a = -1, \quad b = 1, \quad c = -4 \\
\Rightarrow\quad & y_{PI} = -x^2 + x - 4 \\
\Rightarrow\quad & y(x) = C_1 \cosh x + C_2 \sinh x - x^2 + x - 4.
\end{aligned}
$$

6. Proceeding as in **5**, we have

$$\begin{aligned} & s^2 - 2s - 8 = 0 \quad \Rightarrow \quad s_1 = 4, \;\; s_2 = -2 \\ \Rightarrow \quad & y_{CF} = C_1 e^{4x} + C_2 e^{-2x}, \\ & y_{PI} = ax^2 + bx + c = x^2 - x \\ \Rightarrow \quad & y(x) = C_1 e^{4x} + C_1 e^{-2x} + x^2 - x. \end{aligned}$$

7. Proceeding as in **5**, we have

$$\begin{aligned} & s^2 - 25 = 0 \quad \Rightarrow \quad s_{1,2} = \pm 5 \\ \Rightarrow \quad & y_{CF} = C_1 e^{5x} + C_2 e^{-5x}, \\ & y_{PI} = axe^{-5x} \\ \Rightarrow \quad & a(-10e^{-5x} + 25xe^{-5x}) - 25axe^{-5x} = -10ae^{-5x} = 30e^{-5x} \\ \Rightarrow \quad & -10a = 30 \qquad \Rightarrow \quad a = -3 \\ \Rightarrow \quad & y_{PI} = -3xe^{-5x} \\ \Rightarrow \quad & y(x) = C_1 e^{5x} + C_2 e^{-5x} - 3xe^{-5x} = C_1 e^{5x} + (C_2 - 3x)e^{-5x}. \end{aligned}$$

8. Proceeding as in **7**, we have

$$\begin{aligned} & 4s^2 + 1 = 0 \quad \Rightarrow \quad s_{1,2} = \pm \tfrac{1}{2}\, i \\ \Rightarrow \quad & y_{CF} = C_1 \cos\left(\tfrac{1}{2}\, x\right) + C_2 \sin\left(\tfrac{1}{2}\, x\right), \\ & y_{PI} = x\Big[a \cos\left(\tfrac{1}{2}\, x\right) + b \sin\left(\tfrac{1}{2}\, x\right)\Big] = 2x \sin\left(\tfrac{1}{2}\, x\right) \\ \Rightarrow \quad & y(x) = C_1 \cos\left(\tfrac{1}{2}\, x\right) + (C_2 + 2x) \sin\left(\tfrac{1}{2}\, x\right). \end{aligned}$$

Section 1.4

1. This is a Cauchy–Euler equation, so, proceeding as in Example 1.14, we have

$$\begin{aligned} & 2r^2 - r - 3 = 0 \quad \Rightarrow \quad r_1 = \tfrac{3}{2}, \;\; r_2 = -1 \\ \Rightarrow \quad & y(x) = C_1 x^{3/2} + C_2 x^{-1}. \end{aligned}$$

2. Proceeding as in **1**, we have

$$\begin{aligned} & r^2 + r - 6 = 0 \quad \Rightarrow \quad r_1 = 2, \;\; r_2 = -3 \\ \Rightarrow \quad & y(x) = C_1 x^2 + C_2 x^{-3}. \end{aligned}$$

Section 1.5

1. For any functions y_1, y_2 and any numbers c_1, c_2,

$$\begin{aligned} & x(c_1 y_1 + c_2 y_2)'' - (c_1 y_1 + c_2 y_2)' \sin x = c_1(xy_1'' - y_1' \sin x) + c_2(xy_2'' - y_2' \sin x) \\ \Rightarrow \quad & \text{equation is linear.} \end{aligned}$$

2. For functions y_1, y_2 and numbers c_1, c_2, in general we have

$$\begin{aligned} (c_1 y_1 + c_2 y_2)' + 2x \sin(c_1 y_1 + c_2 y_2) &= c_1 y_1' + c_2 y_2' + 2x \sin(c_1 y_1 + c_2 y_2) \\ &\neq c_1(y_1' + 2x \sin y_1) + c_2(y_2' + 2x \sin y_2) \\ &= c_1 y_1' + 2c_1 x \sin y_1 + c_2 y_2' + 2c_2 x \sin y_2 \end{aligned}$$

$$\Rightarrow \quad \text{equation is nonlinear.}$$

3. Proceeding as in **2**, in general we have

$$\begin{aligned}
&(c_1y_1 + c_2y_2)'(c_1y_1 + c_2y_2)'' - x(c_1y_1 + c_2y_2) \\
&\qquad = c_1^2 y_1' y_1'' + c_1c_2(y_1'y_2'' + y_1''y_2') + c_2^2 y_2' y_2'' - c_1(xy_1) - c_2(xy_2) \\
&\qquad \neq c_1(y_1'y_1'' - xy_1) + c_2(y_2'y_2'' - xy_2)
\end{aligned}$$

$\Rightarrow$ equation is nonlinear.

4. Proceeding as in **1**, we have

$$(c_1y_1 + c_2y_2)'' + \sqrt{x}\,(c_1y_1 + c_2y_2) = c_1(y_1'' + \sqrt{x}\,y_1) + c_2(y_2'' + \sqrt{x}\,y_2)$$

$\Rightarrow$ equation is linear.