1 Introduction

1.1 US Energy Production

In 2011, the United States required 3,856 billion kW-hours of electricity. About 20% of this power was generated by ∼100 nuclear fission reactors. About 67% was produced by the burning of fossil fuels, which accounted for about one-third of all greenhouse gas emissions in the U.S. The remaining 13% was generated using other renewable energy resources. Consider the scenario where all the fossil fuel power stations are replaced by new 1-GW nuclear fission reactors. How many such reactors would be needed?

Exercise 1.1 Solution: US Energy Production

$$3.856 \times 10^6 \text{ GW-hours} \times 0.67 \times \frac{\text{years}}{8766 \text{ hours}} \times \frac{\text{Reactor}}{\text{GW}} = 295 \text{ Reactors}$$

1.2 Geothermal Heating

It is estimated that 20 TW of heating in the earth is due to radioactive decay: 8 TW from ^{238}U decay, 8 TW from ^{232}Th decay, and 4 TW from ^{40}K decay. Estimate the total amount of ^{238}U, ^{232}Th, and ^{40}K present in the Earth in order to produce such heating.

Exercise 1.2 Solution: Geothermal Heating

The number of decays in unit time dt is equal to $N(1 - e^{-\mathrm{d}t/\tau}) \approx N\mathrm{d}t/\tau$. That means the total mass of an isotope can be found from:

$$M_{\text{tot.}} = \frac{\tau P}{E_{\text{decay}}} \times M_{\text{atom}}$$

1) ^{238}U

   i) $P = 8 \times 10^{12}$ W $= 4.99 \times 10^{25}$ MeV/s
   ii) $\tau = 6.446 \times 10^9$ years $= 1.64 \times 10^{17}$ s
   iii) $E_{\text{decay}} = 4.267$ MeV
   iv) $M_{\text{atom}} = 238$ amu $= 3.95 \times 10^{-25}$ kg

   $M_{\text{tot.}} = 9.39 \times 10^{17}$ kg

2) ^{232}Th

i) $P = 8 \times 10^{12}$ W $= 4.99 \times 10^{25}$ MeV/s
ii) $\tau = 2.027 \times 10^10$ years $= 6.39 \times 10^{17}$ s
iii) $E_{\text{decay}} = 4.083$ MeV
iv) $M_{\text{atom}} = 232$ amu $= 3.85 \times 10^{-25}$ kg

$M_{\text{tot.}} = 3.00 \times 10^{18}$ kg

3) ^{40}K

i) $P = 4 \times 10^{12}$ W $= 2.50 \times 10^{25}$ MeV/s
ii) $\tau = 1.805 \times 10^{9}$ years $= 5.70 \times 10^{16}$ s
iii) $E_{\text{decay}} = 1.31$ MeV
iv) $M_{\text{atom}} = 40$ amu $= 6.64 \times 10^{-26}$ kg

$M_{\text{tot.}} = 7.23 \times 10^{16}$ kg

1.3 Radioactive Thermoelectric Generators

A useful form of power for space missions which travel far from the sun is a radioactive thermoelectric generator (RTG). Such devices were first suggested by the science fiction writer Arthur C. Clarke in 1945. An RTG uses a thermocouple to convert the heat released by the decay of a radioactive material into electricity by the Seebeck effect. The two Voyager spacecraft have been powered since 1977 by RTGs using ^{238}Pu. Assuming a mass of 5 kg of ^{238}Pu, estimate the heat produced and the electrical power delivered. (Do not forget to include the $\sim 5\%$ thermocouple efficiency.)

Exercise 1.3 Solution: Radioactive Thermoelectric Generators

Let's first look at the instantaneous power produced. We know that:

$$P(t) = \frac{N(t)E_{\text{decay}}}{\tau} = \frac{N(0)e^{-t/\tau}E_{\text{decay}}}{\tau}$$

^{238}Pu has a 5.593 MeV α decay with a lifetime of $\tau = 126.5$ years (3.99×10^{9} s). 5 kg of ^{238}Pu is 1.27×10^{25} atoms. At $t = 0$ years (1977), the thermal power produced was 1.77×10^{16} MeV/s $= 2.84$ kW. At $t = 38$ years (2015), the power was down to 2.10 kW. A typical RTG efficiency is about 5%, so we can estimate that Voyager's power budget has been between 140 and 100 W.

We can integrate the thermal power to get the total heat produced.

$$Q = \int_0^t P(t)\text{d}t = N(0)E_{\text{decay}}\left[1 - e^{-t/\tau}\right]$$

For $t = 38$ years, we get a total heat $Q = 2.95 \times 10^{12}$ J.

1.4 Fission versus Fusion

Energy can be produced by either nuclear fission or nuclear fusion.

a) Consider the fission of ^{235}U into ^{117}Sn and ^{118}Sn, respectively. Using the mass information from a Table of Isotopes, calculate (i) the energy released per fission and (ii) the energy released per atomic mass of fuel.
b) Consider the deuteron-triton fusion reaction

$$^{2}\mathrm{H} + {}^{3}\mathrm{H} \rightarrow {}^{4}\mathrm{He} + \mathrm{n}\ .$$

Using the mass information from the Periodic Table of the Isotopes, calculate (i) the energy released per fusion and (ii) the energy released per atomic mass unit of fuel.

Exercise 1.4 Solution: Fission versus Fusion

Mass defect of ^{235}U: 40.9218 MeV
Mass defect of ^{117}Sn: −90.3977 MeV
Mass defect of ^{118}Sn: −91.6528 MeV

. Per fission:

$$E = (\Delta M_{^{235}U} - \Delta M_{^{117}Sn} - \Delta M_{^{118}Sn}) = 222.97 \text{ MeV}$$

. Per amu:

$$E = 222.97 \text{ MeV/atom} \times \frac{\text{atom}}{235 \text{ amu}} = 0.949 \text{ MeV/amu}$$

Mass defect of ^{2}H: 13.1357 MeV
Mass defect of ^{3}H: 14.9498 MeV
Mass defect of ^{4}He: 2.4249 MeV
Mass defect of n: 8.0713 MeV

. Per fission:

$$E = (\Delta M_{^{2}H} + \Delta M_{^{3}H} - \Delta M_{^{4}He} - \Delta M_{n}) = 17.59 \text{ MeV}$$

. Per amu:

$$E = 17.59 \text{ MeV/fission} \times \frac{\text{fission}}{5 \text{ amu}} = 3.518 \text{ MeV/amu}$$

1.5 Absorption Lengths

A flux of particles is incident upon a thick layer of absorbing material. Find the absorption length, the distance after which the particle intensity is reduced by a factor of $1/e \sim 37\%$ (the absorption length) for each of the following cases:

a) when the particles are thermal neutrons (*i.e.,* neutrons having thermal energies), the absorber is cadmium, and the cross section is 24,500 barns,
b) when the particles are 2 MeV photons, the absorber is lead, and the cross section is 15.7 barns per atom,

c) when the particles are anti-neutrinos from a reactor, the absorber is the Earth, and the cross section is 10^{-19} barns per atomic electron.

Exercise 1.5 Solution: Absorption Lengths

The beam intensity will fall exponentially, according to the differential equation:

$$\frac{dN}{dx} = -\frac{N}{\lambda}$$

where λ is the absorbtion length. The beam is reduced by factor of $1/e$ when $x = \lambda$. The reduction of beam intensity is proportional to the number of interactions per distance, i.e. σn, where σ is the interaction cross section and n is the number density in the absorber.

$$\lambda = -\frac{N}{\frac{dN}{dx}} = \frac{1}{\sigma n}$$

In each part of this problem, one must find a plausible value for n and solve for λ.

1. Neutrons in Cadmium:
   The density of cadmium is 8.65 g/cm^3. The atomic weight of cadmium is 112 amu $= 1.86 \times 10^{-22}$ g, making $n = 4.65 \times 10^{22}$ cm^{-3}.

$$\lambda = \frac{1}{[4.65 \times 10^{22}\ \text{cm}^{-3}]\,[24500\ \text{b}]} \times \frac{\text{b}}{10^{-24}\ \text{cm}} = 8.78 \times 10^{-4}\ \text{cm} = 8.78\mu\text{m}$$

2. Photons in Lead:
   The density of lead is 11.3 g/cm^3. The atomic weight of lead is 207 amu $= 3.44 \times 10^{-22}$ g, making $n = 3.28 \times 10^{22}$ cm^{-3}.

$$\lambda = \frac{1}{[3.28 \times 10^{22}\ \text{cm}^{-3}]\,[15.7\ \text{b}]} \times \frac{\text{b}}{10^{-24}\ \text{cm}} = 1.94\ \text{cm}$$

3. Anti-neutrinos through the Earth:
   The average density of the earth is 5.51 g/cm^3. The four most abundant elements on earth, oxygen, magnesium, silicon, and iron, make up more than 93% of the earth's mass. A mass-weighted average of their values of e per amu comes to 0.488 e/amu $=$ 2.94 $\times 10^{23} e$/g. The electron density of the earth is thus $1.51 \times 10^{24} e/\text{cm}^3$.

$$\lambda = \frac{1}{[1.51 \times 10^{24}\ \text{cm}^{-3}]\,[10^{-19}\ \text{b}]} \times \frac{\text{b}}{10^{-24}\ \text{cm}} = 6.62 \times 10^{18}\ \text{cm} \approx 7\ \text{light-years}$$

2 Symmetries

2.1 Noether's Theorem

The mathematician Emmy Noether proved a very important theorem to physics which states that for any invariance of the classical action under a continuous field transformation there exists a classical charge Q which is independent of time and which is connected to a conserved current, $\partial_\mu J^\mu = 0$. That is, the existence of a a symmetry requires the validity of a corresponding conservation law. Well known examples in classical physics are the invariance of the equations of motion under spatial translation, time translation, and rotation, which lead to conservation of momentum, energy, and angular momentum respectively. The purpose of this exercise is to prove this theorem.

We begin with an action

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

and consider an infinitesimal transformation of the field

$$\phi \to \phi' = \phi + \epsilon f(\phi)\,,.$$

where $f(\phi)$ is some function of the field.

a) Find the equation of motion for a constant value of the infinitesimal ϵ and show that one finds the Euler-Lagrange equation

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = 0\ .$$

b) Calculate the change of the action under this field transformation in the case that $\epsilon = \epsilon(x)$ and show that

$$S \to S' = S + \int d^4x \partial_\mu \epsilon j^\mu\ ,$$

where

$$j^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} f(\phi)\ .$$

c) Now integrate by parts and show that if the action is invariant we require that $\partial_\mu j^\mu = 0$, up to a total derivative.

d) Integrate the equation

$$\frac{\partial j^0}{\partial t} = -\nabla \cdot \mathbf{j} = 0$$

over all space, and show that for a local "charge" distribution we have

$$\frac{dQ}{dt} = 0 \qquad \text{where} \quad Q = \int d\mathbf{x} j^0 \ ,$$

which proves Noether's theorem.

Exercise 2.1 Solution: Noether's Theorem

a) We have

$$0 = \delta S = \int d^4x \left[\frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\delta\partial_\mu\phi\right] = \int d^4x \epsilon f(\phi)\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\right]$$

so

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = 0$$

b) Now

$$0 = \delta S = \int d^4x \left[\delta\mathcal{L}\delta\phi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\delta\partial_\mu\phi\right]$$

$$= \int d^4x \left[\epsilon f(\phi)\frac{\delta \mathcal{L}}{\delta \phi} + (f(\phi)\partial_\mu\epsilon + \epsilon\partial_\mu f(\phi))\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\right]$$

$$= \int d^4x \epsilon f(\phi)\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\right] + \partial_\mu \epsilon f(\phi)\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}$$

c) Then, using, result of a), we have

$$0 = \int d^4x \partial_\mu \epsilon f(\phi)\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} = \int d^4x \epsilon \partial^\mu j_\mu$$

with

$$j_\mu = f(\phi)\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}$$

d) Since

$$0 = \frac{\delta \rho}{\delta t} + \nabla \cdot \mathbf{j}$$

we have

$$\frac{d}{dt}\int d^3x \rho = -\int d^3x \nabla \cdot \mathbf{j} = \int d\mathbf{A} \cdot \mathbf{j} \overset{R\to\infty}{\longrightarrow} 0$$

2.2 Rotation Matrices and Finite Rotations

The effect on a spin eigenstate $|S,S_z\rangle$ under a rotation by angle χ about an axis $\hat{\mathbf{n}}$ is given by

$$|S,S_z\rangle' = R^S(\hat{\mathbf{n}},\chi)\,|S,S_z\rangle \ ,$$

where

$$R^S(\hat{\mathbf{n}},\chi) = \exp(-i\chi\mathbf{S}\cdot\hat{\mathbf{n}}) \ ,$$

and where $\mathbf{S}$ are the $(2S+1)\times(2S+1)$ component spin matrices constructed from the relations

$$S_z\,|S,m\rangle = m\,|S,m\rangle$$

$$(S_x \pm iS_y)\,|S,m\rangle = \sqrt{(S\mp m)(S\pm m+1)}\,|S,m\pm 1\rangle \ .$$

a) Evaluate the rotation matrix for a spin-$\frac{1}{2}$ system and show that

$$R^{\frac{1}{2}}(\hat{\mathbf{n}},\chi) = \exp(-i\chi\mathbf{S}^{\frac{1}{2}}\cdot\hat{\mathbf{n}}) = \cos\frac{\chi}{2} - i\sigma\cdot\hat{\mathbf{n}}\sin\frac{\chi}{2} \ ,$$

where σ are the Pauli matrices.

b) Calculate the rotated spin-$\frac{1}{2}$ state $|1/2,m\rangle'$ for the initial state $|1/2,m\rangle$ for $m = \pm\frac{1}{2}$ using $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$ and rotation angle χ and demonstrate that

$$|1/2,m\rangle' = \exp(-im\chi)\,|1/2,m\rangle \ .$$

c) Verify the commutation relations for the representations of the spin-1 operators in Eq. (2.41),

$$[S_z,S_\pm] = \pm S_\pm$$

$$[S_+,S_-] = 2S_z \ .$$

d) Now, evaluate the rotation matrix for a spin-1 system and show that

$$R^1(\hat{\mathbf{n}},\chi) = \exp(-i\chi\mathbf{S}^1\cdot\hat{\mathbf{n}}) = 1 - (\mathbf{S}^1\cdot\hat{\mathbf{n}})^2(1-\cos\chi) - i\mathbf{S}^1\cdot\hat{\mathbf{n}}\sin\chi \ ,$$

where $\mathbf{S}^1$ are the 3×3 spin matrices constructed in the text.

e) Calculate the rotated spin state $|1,m\rangle'$ for the initial state $|1,m\rangle$ for the cases $m = 1,0,-1$ using $\hat{\mathbf{n}} = \hat{\mathbf{e}}_z$ and rotation angle χ and demonstrate that

$$|1,m\rangle' = \exp(-im\chi)\,|1,m\rangle \,.$$

Exercise 2.2 Solution: Rotation Matrices and Finite Rotations

a) We have $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}^2 = 1$ so

$$\exp -i\frac{\chi}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(-\frac{\chi}{2}\right)^{2n} + i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(-\frac{\chi}{2}\right)^{2n+1}$$

$$= \cos\frac{\chi}{2} - i\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\sin\frac{\chi}{2}$$

b) Pick $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ so

$$R^{\frac{1}{2}}(\chi, \hat{\mathbf{n}})|\frac{1}{2}, \pm\frac{1}{2} >= \left(\cos\frac{\chi}{2} - i\sigma_z \sin\frac{\chi}{2}\right)|\frac{1}{2}, \pm\frac{1}{2} >$$

$$= \left(\cos\frac{\chi}{2} \mp i\sin\frac{\chi}{2}\right)|\frac{1}{2}, \pm\frac{1}{2} >= \exp\left(\mp i\frac{\chi}{2}\right)|\frac{1}{2}, \pm\frac{1}{2} >$$

c) Have

$$S_z S_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$S_z S_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$[S_z, S_+] = S_z S_+ - S_+ S_z = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} = S_+$$

Also, we have

$$S_+ S_- = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$S_- S_+ = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

so

$$[S_+, S_-] = S_+ S_- - S_- S_+ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2S_z$$

d) Since $(\mathbf{S}\cdot\hat{\mathbf{n}})^3 = \mathbf{S}\cdot\hat{\mathbf{n}}$ we have

$$\exp -i\chi\mathbf{S}\cdot\hat{\mathbf{n}} = \mathbf{1} + (\mathbf{S}\cdot\hat{\mathbf{n}})^2 \sum_{n=1}^{\infty} \frac{1}{(2n)!}(-\chi)^{2n} + i\mathbf{S}\cdot\hat{\mathbf{n}} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-\chi)^{2n+1}$$

$$= \mathbf{1} + (\mathbf{S}\cdot\hat{\mathbf{n}})^2(\cos\chi - 1) - i\mathbf{S}\cdot\hat{\mathbf{n}}\sin\chi$$

e) Pick $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ so

$$R^1(\chi, \hat{\mathbf{n}})|1, m >= \left(\mathbf{1} - S_z^2(1-\cos\chi) - iS_z\sin\chi\right)|1, m >$$