Chapter 1

The Wave Function

Problem 1.1

(a)

$$\langle j\rangle^2 = 21^2 = \boxed{441.}$$

$$\begin{aligned}\langle j^2\rangle &= \frac{1}{N}\sum j^2 N(j) = \frac{1}{14}\left[(14^2) + (15^2) + 3(16^2) + 2(22^2) + 2(24^2) + 5(25^2)\right] \\ &= \frac{1}{14}(196 + 225 + 768 + 968 + 1152 + 3125) = \frac{6434}{14} = \boxed{459.571.}\end{aligned}$$

(b)

j	$\Delta j = j - \langle j\rangle$
14	$14 - 21 = -7$
15	$15 - 21 = -6$
16	$16 - 21 = -5$
22	$22 - 21 = 1$
24	$24 - 21 = 3$
25	$25 - 21 = 4$

$$\begin{aligned}\sigma^2 &= \frac{1}{N}\sum(\Delta j)^2 N(j) = \frac{1}{14}\left[(-7)^2 + (-6)^2 + (-5)^2\cdot 3 + (1)^2\cdot 2 + (3)^2\cdot 2 + (4)^2\cdot 5\right] \\ &= \frac{1}{14}(49 + 36 + 75 + 2 + 18 + 80) = \frac{260}{14} = \boxed{18.571.}\end{aligned}$$

$$\sigma = \sqrt{18.571} = \boxed{4.309.}$$

(c)

$$\langle j^2\rangle - \langle j\rangle^2 = 459.571 - 441 = 18.571. \quad \text{[Agrees with (b).]}$$

Problem 1.2

(a)

$$\langle x^2\rangle = \int_0^h x^2 \frac{1}{2\sqrt{hx}}dx = \frac{1}{2\sqrt{h}}\left(\frac{2}{5}x^{5/2}\right)\Big|_0^h = \frac{h^2}{5}.$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{h^2}{5} - \left(\frac{h}{3}\right)^2 = \frac{4}{45}h^2 \;\Rightarrow\; \sigma = \boxed{\frac{2h}{3\sqrt{5}} = 0.2981h.}$$

(b)

$$P = 1 - \int_{x_-}^{x_+} \frac{1}{2\sqrt{hx}}dx = 1 - \frac{1}{2\sqrt{h}}(2\sqrt{x})\Big|_{x_-}^{x_+} = 1 - \frac{1}{\sqrt{h}}\left(\sqrt{x_+} - \sqrt{x_-}\right).$$

$$x_+ \equiv \langle x\rangle + \sigma = 0.3333h + 0.2981h = 0.6315h; \quad x_- \equiv \langle x\rangle - \sigma = 0.3333h - 0.2981h = 0.0352h.$$

$$P = 1 - \sqrt{0.6315} + \sqrt{0.0352} = \boxed{0.393.}$$

Problem 1.3

(a)

$$1 = \int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}dx. \qquad \text{Let } u \equiv x - a,\ du = dx,\ u : -\infty \to \infty.$$

$$1 = A\int_{-\infty}^{\infty} e^{-\lambda u^2}du = A\sqrt{\frac{\pi}{\lambda}} \quad \Rightarrow \quad \boxed{A = \sqrt{\frac{\lambda}{\pi}}.}$$

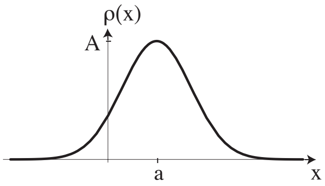
(b)

$$\begin{aligned}\langle x\rangle &= A\int_{-\infty}^{\infty} xe^{-\lambda(x-a)^2}dx = A\int_{-\infty}^{\infty}(u+a)e^{-\lambda u^2}du \\ &= A\left[\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right] = A\left(0 + a\sqrt{\frac{\pi}{\lambda}}\right) = \boxed{a.}\end{aligned}$$

$$\begin{aligned}\langle x^2\rangle &= A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}dx \\ &= A\left\{\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}du + 2a\int_{-\infty}^{\infty} ue^{-\lambda u^2}du + a^2\int_{-\infty}^{\infty} e^{-\lambda u^2}du\right\} \\ &= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right] = \boxed{a^2 + \frac{1}{2\lambda}.}\end{aligned}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2 + \frac{1}{2\lambda} - a^2 = \frac{1}{2\lambda}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2\lambda}}.}$$

(c)



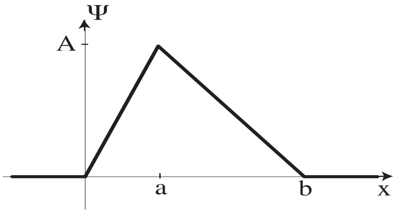
Problem 1.4

(a)

$$1 = \frac{|A|^2}{a^2}\int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2}\int_a^b (b-x)^2 dx = |A|^2\left\{\frac{1}{a^2}\left(\frac{x^3}{3}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(-\frac{(b-x)^3}{3}\right)\Big|_a^b\right\}$$

$$= |A|^2\left[\frac{a}{3} + \frac{b-a}{3}\right] = |A|^2\frac{b}{3} \;\Rightarrow\; \boxed{A = \sqrt{\frac{3}{b}}.}$$

(b)



(c) At $\boxed{x = a.}$

(d)

$$P = \int_0^a |\Psi|^2 dx = \frac{|A|^2}{a^2}\int_0^a x^2 dx = |A|^2\frac{a}{3} = \boxed{\frac{a}{b}.}\begin{cases} P = 1 & \text{if } b = a, \ \checkmark \\ P = 1/2 & \text{if } b = 2a. \ \checkmark \end{cases}$$

(e)

$$\langle x\rangle = \int x|\Psi|^2 dx = |A|^2\left\{\frac{1}{a^2}\int_0^a x^3 dx + \frac{1}{(b-a)^2}\int_a^b x(b-x)^2 dx\right\}$$

$$= \frac{3}{b}\left\{\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right\}$$

$$= \frac{3}{4b(b-a)^2}\left[a^2(b-a)^2 + 2b^4 - 8b^4/3 + b^4 - 2a^2b^2 + 8a^3b/3 - a^4\right]$$

$$= \frac{3}{4b(b-a)^2}\left(\frac{b^4}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4(b-a)^2}(b^3 - 3a^2b + 2a^3) = \boxed{\frac{2a+b}{4}.}$$

Problem 1.5

(a)

$$1 = \int |\Psi|^2 dx = 2|A|^2 \int_0^\infty e^{-2\lambda x} dx = 2|A|^2 \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Big|_0^\infty = \frac{|A|^2}{\lambda}; \quad \boxed{A = \sqrt{\lambda}.}$$

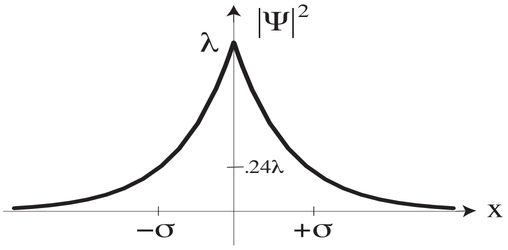
(b)

$$\langle x \rangle = \int x|\Psi|^2 dx = |A|^2 \int_{-\infty}^{\infty} x e^{-2\lambda|x|} dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2 \rangle = 2|A|^2 \int_0^\infty x^2 e^{-2\lambda x} dx = 2\lambda \left[\frac{2}{(2\lambda)^3} \right] = \boxed{\frac{1}{2\lambda^2}.}$$

(c)

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2\lambda^2}; \qquad \boxed{\sigma = \frac{1}{\sqrt{2}\lambda}.} \qquad |\Psi(\pm\sigma)|^2 = |A|^2 e^{-2\lambda\sigma} = \lambda e^{-2\lambda/\sqrt{2}\lambda} = \lambda e^{-\sqrt{2}} = 0.2431\lambda.$$



Probability outside:

$$2\int_\sigma^\infty |\Psi|^2 dx = 2|A|^2 \int_\sigma^\infty e^{-2\lambda x} dx = 2\lambda \left(\frac{e^{-2\lambda x}}{-2\lambda} \right)\Big|_\sigma^\infty = e^{-2\lambda\sigma} = \boxed{e^{-\sqrt{2}} = 0.2431.}$$

Problem 1.6

For integration by parts, the differentiation has to be with respect to the *integration* variable – in this case the differentiation is with respect to t, but the integration variable is x. It's true that

$$\frac{\partial}{\partial t}(x|\Psi|^2) = \frac{\partial x}{\partial t}|\Psi|^2 + x\frac{\partial}{\partial t}|\Psi|^2 = x\frac{\partial}{\partial t}|\Psi|^2,$$

but this does *not* allow us to perform the integration:

$$\int_a^b x\frac{\partial}{\partial t}|\Psi|^2 dx = \int_a^b \frac{\partial}{\partial t}(x|\Psi|^2)dx \neq (x|\Psi|^2)\Big|_a^b.$$

Problem 1.7

From Eq. 1.33, $\frac{d\langle p\rangle}{dt} = -i\hbar\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right)dx$. But, noting that $\frac{\partial^2\Psi}{\partial x\partial t} = \frac{\partial^2\Psi}{\partial t\partial x}$ and using Eqs. 1.23-1.24:

$$\frac{\partial}{\partial t}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) = \frac{\partial\Psi^*}{\partial t}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial\Psi}{\partial t}\right) = \left[-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]$$

$$= \frac{i\hbar}{2m}\left[\Psi^*\frac{\partial^3\Psi}{\partial x^3} - \frac{\partial^2\Psi^*}{\partial x^2}\frac{\partial\Psi}{\partial x}\right] + \frac{i}{\hbar}\left[V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial}{\partial x}(V\Psi)\right]$$

The first term integrates to zero, using integration by parts twice, and the second term can be simplified to $V\Psi^*\frac{\partial\Psi}{\partial x} - \Psi^*V\frac{\partial\Psi}{\partial x} - \Psi^*\frac{\partial V}{\partial x}\Psi = -|\Psi|^2\frac{\partial V}{\partial x}$. So

$$\frac{d\langle p\rangle}{dt} = -i\hbar\left(\frac{i}{\hbar}\right)\int -|\Psi|^2\frac{\partial V}{\partial x}dx = \langle -\frac{\partial V}{\partial x}\rangle. \qquad \text{QED}$$

Problem 1.8

Suppose Ψ satisfies the Schrödinger equation *without* V_0: $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. We want to find the solution Ψ_0 *with* V_0: $i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0$.

Claim: $\Psi_0 = \Psi e^{-iV_0t/\hbar}$.

Proof: $i\hbar\frac{\partial\Psi_0}{\partial t} = i\hbar\frac{\partial\Psi}{\partial t}e^{-iV_0t/\hbar} + i\hbar\Psi\left(-\frac{iV_0}{\hbar}\right)e^{-iV_0t/\hbar} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi\right]e^{-iV_0t/\hbar} + V_0\Psi e^{-iV_0t/\hbar}$
$= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_0}{\partial x^2} + (V+V_0)\Psi_0.$ QED

This has *no* effect on the expectation value of a dynamical variable, since the extra phase factor, being independent of x, cancels out in Eq. 1.36.

Problem 1.9

(a)

$$1 = 2|A|^2\int_0^\infty e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2}\sqrt{\frac{\pi}{(2am/\hbar)}} = |A|^2\sqrt{\frac{\pi\hbar}{2am}}; \qquad \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}.}$$

(b)

$$\frac{\partial\Psi}{\partial t} = -ia\Psi; \quad \frac{\partial\Psi}{\partial x} = -\frac{2amx}{\hbar}\Psi; \quad \frac{\partial^2\Psi}{\partial x^2} = -\frac{2am}{\hbar}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right) = -\frac{2am}{\hbar}\left(1 - \frac{2amx^2}{\hbar}\right)\Psi.$$

Plug these into the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$:

$$V\Psi = i\hbar(-ia)\Psi + \frac{\hbar^2}{2m}\left(-\frac{2am}{\hbar}\right)\left(1 - \frac{2amx^2}{\hbar}\right)\Psi$$

$$= \left[\hbar a - \hbar a\left(1 - \frac{2amx^2}{\hbar}\right)\right]\Psi = 2a^2mx^2\Psi, \quad \text{so} \quad \boxed{V(x) = 2ma^2x^2.}$$

(c)

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi|^2 dx = \boxed{0.} \qquad \text{[Odd integrand.]}$$

$$\langle x^2\rangle = 2|A|^2\int_0^{\infty} x^2 e^{-2amx^2/\hbar}dx = 2|A|^2\frac{1}{2^2(2am/\hbar)}\sqrt{\frac{\pi\hbar}{2am}} = \boxed{\frac{\hbar}{4am}.}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.}$$

$$\begin{aligned}\langle p^2\rangle &= \int \Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2\Psi dx = -\hbar^2\int\Psi^*\frac{\partial^2\Psi}{\partial x^2}dx\\ &= -\hbar^2\int\Psi^*\left[-\frac{2am}{\hbar}\left(1-\frac{2amx^2}{\hbar}\right)\Psi\right]dx = 2am\hbar\left\{\int|\Psi|^2dx - \frac{2am}{\hbar}\int x^2|\Psi|^2dx\right\}\\ &= 2am\hbar\left(1-\frac{2am}{\hbar}\langle x^2\rangle\right) = 2am\hbar\left(1-\frac{2am}{\hbar}\frac{\hbar}{4am}\right) = 2am\hbar\left(\frac{1}{2}\right) = \boxed{am\hbar.}\end{aligned}$$

(d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{4am} \Longrightarrow \boxed{\sigma_x = \sqrt{\frac{\hbar}{4am}}}; \quad \sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = am\hbar \Longrightarrow \boxed{\sigma_p = \sqrt{am\hbar}.}$$

$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{4am}}\sqrt{am\hbar} = \frac{\hbar}{2}$. This *is* (just barely) consistent with the uncertainty principle.

Problem 1.10

From Math Tables: $\pi = 3.141592653589793238462643\cdots$

(a) $\boxed{\begin{array}{lllll} P(0)=0 & P(1)=2/25 & P(2)=3/25 & P(3)=5/25 & P(4)=3/25\\ P(5)=3/25 & P(6)=3/25 & P(7)=1/25 & P(8)=2/25 & P(9)=3/25\end{array}}$

In general, $P(j) = \frac{N(j)}{N}$.

(b) *Most probable*: $\boxed{3.}$ *Median*: 13 are ≤ 4, 12 are ≥ 5, so median is $\boxed{4.}$

Average: $\langle j\rangle = \frac{1}{25}[0\cdot 0 + 1\cdot 2 + 2\cdot 3 + 3\cdot 5 + 4\cdot 3 + 5\cdot 3 + 6\cdot 3 + 7\cdot 1 + 8\cdot 2 + 9\cdot 3]$

$= \frac{1}{25}[0+2+6+15+12+15+18+7+16+27] = \frac{118}{25} = \boxed{4.72.}$

(c) $\langle j^2\rangle = \frac{1}{25}[0 + 1^2\cdot 2 + 2^2\cdot 3 + 3^2\cdot 5 + 4^2\cdot 3 + 5^2\cdot 3 + 6^2\cdot 3 + 7^2\cdot 1 + 8^2\cdot 2 + 9^2\cdot 3]$

$= \frac{1}{25}[0+2+12+45+48+75+108+49+128+243] = \frac{710}{25} = \boxed{28.4.}$

$\sigma^2 = \langle j^2\rangle - \langle j\rangle^2 = 28.4 - 4.72^2 = 28.4 - 22.2784 = 6.1216;\quad \sigma = \sqrt{6.1216} = \boxed{2.474.}$

Problem 1.11

(a)

$$\frac{1}{2}mv^2+V=E \quad \rightarrow \quad \boxed{v(x)=\sqrt{\frac{2}{m}\left(E-V(x)\right)}.}$$

(b)

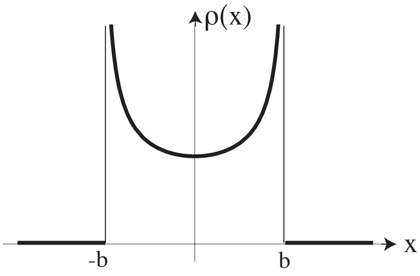
$$T=\int_a^b \frac{1}{\sqrt{\frac{2}{m}\left(E-\frac{1}{2}kx^2\right)}}\,dx=\sqrt{\frac{m}{k}}\int_a^b \frac{1}{\sqrt{(2E/k)-x^2}}\,dx.$$

Turning points: $v=0\Rightarrow E=V=\frac{1}{2}kb^2\Rightarrow b=\sqrt{2E/k};\quad a=-b.$

$$\begin{aligned}T&=2\sqrt{\frac{m}{k}}\int_0^b \frac{1}{\sqrt{b^2-x^2}}\,dx=2\sqrt{\frac{m}{k}}\sin^{-1}\left(\frac{x}{b}\right)\Big|_0^b=2\sqrt{\frac{m}{k}}\sin^{-1}(1)\\&=2\sqrt{\frac{m}{k}}\left(\frac{\pi}{2}\right)=\pi\sqrt{\frac{m}{k}}.\end{aligned}$$

$$\rho(x)=\frac{1}{\pi\sqrt{\frac{m}{k}}\sqrt{\frac{2}{m}\left(E-\frac{1}{2}kx^21\right)}}=\boxed{\frac{1}{\pi\sqrt{b^2-x^2}}.}$$

$$\int_a^b \rho(x)\,dx=\frac{2}{\pi}\int_0^b\frac{1}{\sqrt{b^2-x^2}}\,dx=\frac{2}{\pi}\left(\frac{\pi}{2}\right)=1.\ \checkmark$$



(c) $\boxed{\langle x\rangle=0.}$

$$\begin{aligned}\langle x^2\rangle&=\frac{1}{\pi}\int_{-b}^b\frac{x^2}{\sqrt{b^2-x^2}}\,dx=\frac{2}{\pi}\int_0^b\frac{x^2}{\sqrt{b^2-x^2}}\,dx\\&=\frac{2}{\pi}\left[-\frac{x}{2}\sqrt{b^2-x^2}+\frac{b^2}{2}\sin^{-1}\left(\frac{x}{b}\right)\right]\Big|_0^b=\frac{b^2}{\pi}\sin^{-1}(1)=\frac{b^2}{\pi}\frac{\pi}{2}=\frac{b^2}{2}=\boxed{\frac{E}{k}.}\end{aligned}$$

$$\sigma_x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\langle x^2\rangle}=\frac{b}{\sqrt{2}}=\boxed{\sqrt{\frac{E}{k}}.}$$

Problem 1.12

(a)

$$\rho(p)\,dp=\frac{dt}{T}=\frac{|dt/dp|\,dp}{T}$$

where dt is now the time it spends with momentum in the range dp (dt is intrinsically positive, but $dp/dt=F=-kx$ runs negative—hence the absolute value). Now

$$\frac{p^2}{2m}+\frac{1}{2}kx^2=E\Rightarrow x=\pm\sqrt{\frac{2}{k}\left(E-\frac{p^2}{2m}\right)},$$

so

$$\rho(p) = \frac{1}{\pi\sqrt{\frac{m}{k}}\,k\sqrt{\frac{2}{k}\left(E-\frac{p^2}{2m}\right)}} = \boxed{\frac{1}{\pi\sqrt{2mE-p^2}}} = \frac{1}{\pi\sqrt{c^2-p^2}},$$

where $c \equiv \sqrt{2mE}$. This is the same as $\rho(x)$ (Problem 1.11(b)), with c in place of b (and, of course, p in place of x).

(b) From Problem 1.11(c), $\boxed{\langle p\rangle = 0,\ \langle p^2\rangle = \frac{c^2}{2},\ \sigma_p = \frac{c}{\sqrt{2}} = \sqrt{mE}\,.}$

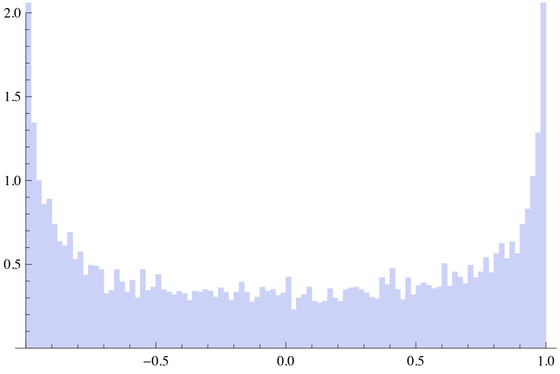
(c) $\sigma_x\sigma_p = \sqrt{\frac{E}{k}}\sqrt{mE} = \boxed{\sqrt{\frac{m}{k}}\,E} = \frac{E}{\omega}$. If $E \geq \frac{1}{2}\hbar\omega$, then $\sigma_x\sigma_p \geq \frac{1}{2}\hbar$, which is precisely the Heisenberg uncertainty principle!

Problem 1.13

```
x[t_] := Cos[t]

snapshots = Table[x[π RandomReal[j]], {j, 10 000}]
```

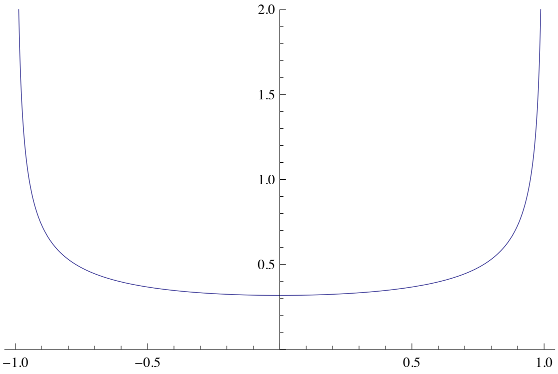
```
Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}]
```



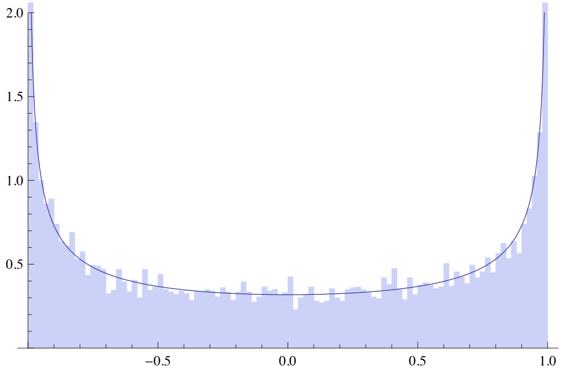
```
r[x_] :=
```

$$\frac{1}{\pi\sqrt{1 - x^2}}$$

```
Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]
```



```
Show[Histogram[snapshots, 100, "PDF", PlotRange → {0, 2}],
 Plot[ r[x], {x, -1, 1}, PlotRange -> {0, 2}]]
```



Problem 1.14

(a) $P_{ab}(t) = \int_a^b |\Psi(x,t)|^2 dx$, so $\frac{dP_{ab}}{dt} = \int_a^b \frac{\partial}{\partial t}|\Psi|^2 dx$. But (Eq. 1.25):

$$\frac{\partial|\Psi|^2}{\partial t} = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right] = -\frac{\partial}{\partial x}J(x,t).$$

$$\therefore \frac{dP_{ab}}{dt} = -\int_a^b \frac{\partial}{\partial x}J(x,t)dx = -\left[J(x,t)\right]\Big|_a^b = J(a,t) - J(b,t). \qquad \text{QED}$$

Probability is dimensionless, so J has the dimensions 1/time, and units $\boxed{\text{seconds}^{-1}.}$

(b) Here $\Psi(x,t) = f(x)e^{-iat}$, where $f(x) \equiv Ae^{-amx^2/\hbar}$, so $\Psi\frac{\partial\Psi^*}{\partial x} = fe^{-iat}\frac{df}{dx}e^{iat} = f\frac{df}{dx}$, and $\Psi^*\frac{\partial\Psi}{\partial x} = f\frac{df}{dx}$ too, so $\boxed{J(x,t) = 0.}$

Problem 1.15

Use Eqs. [1.23] and [1.24], and integration by parts:

$$\begin{aligned}
\frac{d}{dt}\int_{-\infty}^{\infty}\Psi_1^*\Psi_2\,dx &= \int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left(\Psi_1^*\Psi_2\right)dx = \int_{-\infty}^{\infty}\left(\frac{\partial\Psi_1^*}{\partial t}\Psi_2+\Psi_1^*\frac{\partial\Psi_2}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty}\left[\left(\frac{-i\hbar}{2m}\frac{\partial^2\Psi_1^*}{\partial x^2}+\frac{i}{\hbar}V\Psi_1^*\right)\Psi_2+\Psi_1^*\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi_2}{\partial x^2}-\frac{i}{\hbar}V\Psi_2\right)\right]dx \\
&= -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\frac{\partial^2\Psi_1^*}{\partial x^2}\Psi_2-\Psi_1^*\frac{\partial^2\Psi_2}{\partial x^2}\right)dx \\
&= -\frac{i\hbar}{2m}\left[\frac{\partial\Psi_1^*}{\partial x}\Psi_2\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx-\Psi_1^*\frac{\partial\Psi_2}{\partial x}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{\partial\Psi_1^*}{\partial x}\frac{\partial\Psi_2}{\partial x}dx\right]=0.\ \text{QED}
\end{aligned}$$

Problem 1.16

(a)

$$\begin{aligned}
1 &= |A|^2\int_{-a}^{a}\left(a^2-x^2\right)^2dx = 2|A|^2\int_0^a\left(a^4-2a^2x^2+x^4\right)dx = 2|A|^2\left[a^4x-2a^2\frac{x^3}{3}+\frac{x^5}{5}\right]\Big|_0^a \\
&= 2|A|^2a^5\left(1-\frac{2}{3}+\frac{1}{5}\right)=\frac{16}{15}a^5|A|^2,\ \text{so}\ \boxed{A=\sqrt{\frac{15}{16a^5}}.}
\end{aligned}$$

(b)

$$\langle x\rangle = \int_{-a}^{a}x|\Psi|^2\,dx = \boxed{0.}\quad \text{(Odd integrand.)}$$

(c)

$$\langle p\rangle = \frac{\hbar}{i}A^2\int_{-a}^{a}\left(a^2-x^2\right)\underbrace{\frac{d}{dx}\left(a^2-x^2\right)}_{-2x}dx = \boxed{0.}\quad \text{(Odd integrand.)}$$

Since we only know $\langle x\rangle$ at $t=0$ we cannot calculate $d\langle x\rangle/dt$ directly.

(d)

$$\begin{aligned}
\langle x^2\rangle &= A^2\int_{-a}^{a}x^2\left(a^2-x^2\right)^2dx = 2A^2\int_0^a\left(a^4x^2-2a^2x^4+x^6\right)dx \\
&= 2\frac{15}{16a^5}\left[a^4\frac{x^3}{3}-2a^2\frac{x^5}{5}+\frac{x^7}{7}\right]\Big|_0^a = \frac{15}{8a^5}\left(a^7\right)\left(\frac{1}{3}-\frac{2}{5}+\frac{1}{7}\right) \\
&= \frac{\cancel{15}a^2}{8}\left(\frac{35-42+15}{\cancel{3}\cdot\cancel{5}\cdot 7}\right)=\frac{a^2}{8}\cdot\frac{8}{7}=\boxed{\frac{a^2}{7}.}
\end{aligned}$$

(e)

$$\langle p^2\rangle = -A^2\hbar^2\int_{-a}^{a}\left(a^2-x^2\right)\underbrace{\frac{d^2}{dx^2}\left(a^2-x^2\right)}_{-2}dx = 2A^2\hbar^2 2\int_0^a\left(a^2-x^2\right)dx$$

$$= 4\cdot\frac{15}{16a^5}\hbar^2\left(a^2x-\frac{x^3}{3}\right)\Big|_0^a = \frac{15\hbar^2}{4a^5}\left(a^3-\frac{a^3}{3}\right) = \frac{15\hbar^2}{4a^2}\cdot\frac{2}{3} = \boxed{\frac{5}{2}\frac{\hbar^2}{a^2}.}$$

(f)

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{7}a^2} = \boxed{\frac{a}{\sqrt{7}}.}$$

(g)

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}\frac{\hbar^2}{a^2}} = \boxed{\sqrt{\frac{5}{2}}\frac{\hbar}{a}.}$$

(h)

$$\sigma_x\sigma_p = \frac{a}{\sqrt{7}}\cdot\sqrt{\frac{5}{2}}\frac{\hbar}{a} = \sqrt{\frac{5}{14}}\hbar = \sqrt{\frac{10}{7}}\frac{\hbar}{2} > \frac{\hbar}{2}. \checkmark$$

Problem 1.17

(a) Eq. 1.24 now reads $\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V^*\Psi^*$, and Eq. 1.25 picks up an extra term:

$$\frac{\partial}{\partial t}|\Psi|^2 = \cdots + \frac{i}{\hbar}|\Psi|^2(V^*-V) = \cdots + \frac{i}{\hbar}|\Psi|^2(V_0+i\Gamma-V_0+i\Gamma) = \cdots - \frac{2\Gamma}{\hbar}|\Psi|^2,$$

and Eq. 1.27 becomes $\frac{dP}{dt} = -\frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty}|\Psi|^2dx = -\frac{2\Gamma}{\hbar}P.$ QED

(b)

$$\frac{dP}{P} = -\frac{2\Gamma}{\hbar}dt \Longrightarrow \ln P = -\frac{2\Gamma}{\hbar}t + \text{constant} \Longrightarrow \boxed{P(t) = P(0)e^{-2\Gamma t/\hbar},}\ \text{so}\ \boxed{\tau = \frac{\hbar}{2\Gamma}.}$$

Problem 1.18

$$\frac{h}{\sqrt{3mk_BT}} > d \;\Rightarrow\; T < \frac{h^2}{3mk_Bd^2}.$$

(a) Electrons ($m = 9.1\times10^{-31}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(9.1\times10^{-31})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{1.3\times10^5\ \text{K.}}$$

Silicon nuclei ($m = 28m_p = 28(1.7\times10^{-27}) = 4.8\times10^{-26}$ kg):

$$T < \frac{(6.6\times10^{-34})^2}{3(4.8\times10^{-26})(1.4\times10^{-23})(3\times10^{-10})^2} = \boxed{2.4\ \text{K.}}$$

(b) $PV = Nk_BT$; volume occupied by one molecule $(N = 1,\ V = d^3) \Rightarrow d = (k_BT/P)^{1/3}$.

$$T < \frac{h^2}{3mk_B}\left(\frac{P}{k_BT}\right)^{2/3} \Rightarrow T^{5/3} < \frac{h^2}{3m}\frac{P^{2/3}}{k_B^{5/3}} \Rightarrow T < \frac{1}{k_B}\left(\frac{h^2}{3m}\right)^{3/5} P^{2/5}.$$

For helium $(m = 4m_p = 6.8 \times 10^{-27}$ kg) at 1 atm $= 1.0 \times 10^5$ N/m^2:

$$T < \frac{1}{(1.4 \times 10^{-23})}\left(\frac{(6.6 \times 10^{-34})^2}{3(6.8 \times 10^{-27})}\right)^{3/5} (1.0 \times 10^5)^{2/5} = \boxed{2.8 \text{ K.}}$$

For atomic hydrogen $(m = m_p = 1.7 \times 10^{-27}$ kg) with $d = 0.01$ m:

$$T < \frac{(6.6 \times 10^{-34})^2}{3(1.7 \times 10^{-27})(1.4 \times 10^{-23})(10^{-2})^2} = \boxed{6.2 \times 10^{-14} \text{ K.}}$$

At 3 K it is definitely in the classical regime.

Chapter 2

The Time-Independent Schrödinger Equation

Problem 2.1

(a)

$$\Psi(x,t)=\psi(x)e^{-i(E_0+i\Gamma)t/\hbar}=\psi(x)e^{\Gamma t/\hbar}e^{-iE_0t/\hbar}\Longrightarrow|\Psi|^2=|\psi|^2e^{2\Gamma t/\hbar}.$$

$$\int_{-\infty}^{\infty}|\Psi(x,t)|^2dx=e^{2\Gamma t/\hbar}\int_{-\infty}^{\infty}|\psi|^2dx.$$

The second term is independent of t, so if the product is to be 1 for all time, the first term ($e^{2\Gamma t/\hbar}$) must also be constant, and hence $\Gamma=0$. QED

(b) If ψ satisfies Eq. 2.5, $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi$, then (taking the complex conjugate and noting that V and E are real): $-\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2}+V\psi^*=E\psi^*$, so ψ^* *also* satisfies Eq. 2.5. Now, if ψ_1 and ψ_2 satisfy Eq. 2.5, so too does any linear combination of them ($\psi_3\equiv c_1\psi_1+c_2\psi_2$):

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2}+V\psi_3&=-\frac{\hbar^2}{2m}\left(c_1\frac{d^2\psi_1}{dx^2}+c_2\frac{d^2\psi_2}{dx^2}\right)+V(c_1\psi_1+c_2\psi_2)\\&=c_1\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2}+V\psi_1\right]+c_2\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2}+V\psi_2\right]\\&=c_1(E\psi_1)+c_2(E\psi_2)=E(c_1\psi_1+c_2\psi_2)=E\psi_3.\end{aligned}$$

Thus, $(\psi+\psi^*)$ and $i(\psi-\psi^*)$ – both of which are *real* – satisfy Eq. 2.5. *Conclusion:* From any complex solution, we can always construct two *real* solutions (of course, if ψ is already real, the second one will be zero). In particular, since $\psi=\frac{1}{2}[(\psi+\psi^*)-i(i(\psi-\psi^*))]$, ψ can be expressed as a linear combination of two real solutions. QED

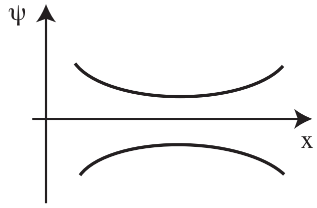
(c) If $\psi(x)$ satisfies Eq. 2.5, then, changing variables $x\to -x$ and noting that $d^2/d(-x)^2=d^2/dx^2$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(-x)}{dx^2}+V(-x)\psi(-x)=E\psi(-x);$$

so if $V(-x)=V(x)$ then $\psi(-x)$ *also* satisfies Eq. 2.5. It follows that $\psi_+(x)\equiv\psi(x)+\psi(-x)$ (which is *even*: $\psi_+(-x)=\psi_+(x)$) and $\psi_-(x)\equiv\psi(x)-\psi(-x)$ (which is *odd*: $\psi_-(-x)=-\psi_-(x)$) both satisfy Eq. 2.5. But $\psi(x)=\frac{1}{2}(\psi_+(x)+\psi_-(x))$, so any solution can be expressed as a linear combination of even and odd solutions. QED

Problem 2.2

Given $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi$, if $E < V_{\min}$, then ψ'' and ψ always have the same sign: If ψ is positive(negative), then ψ'' is also positive(negative). This means that ψ always curves away from the axis (see Figure). However, it has got to go to zero as $x \to -\infty$ (else it would not be normalizable). At some point it's got to *depart* from zero (if it *doesn't*, it's going to be identically zero *everywhere*), in (say) the positive direction. At this point its slope is positive, and *increasing*, so ψ gets bigger and bigger as x increases. It can't ever "turn over" and head back toward the axis, because that would require a negative second derivative—it always has to bend away from the axis. By the same token, if it starts out heading negative, it just runs more and more negative. In neither case is there any way for it to come back to zero, as it must (at $x \to \infty$) in order to be normalizable. QED



Problem 2.3

Equation 2.23 says $\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$; Eq. 2.26 says $\psi(0) = \psi(a) = 0$. If $E = 0$, $d^2\psi/dx^2 = 0$, so $\psi(x) = A + Bx$; $\psi(0) = A = 0 \Rightarrow \psi = Bx$; $\psi(a) = Ba = 0 \Rightarrow B = 0$, so $\psi = 0$. If $E < 0$, $d^2\psi/dx^2 = \kappa^2\psi$, with $\kappa \equiv \sqrt{-2mE}/\hbar$ real, so $\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$. This time $\psi(0) = A + B = 0 \Rightarrow B = -A$, so $\psi = A(e^{\kappa x} - e^{-\kappa x})$, while $\psi(a) = A\left(e^{\kappa a} - e^{-\kappa a}\right) = 0 \Rightarrow$ either $A = 0$, so $\psi = 0$, or else $e^{\kappa a} = e^{-\kappa a}$, so $e^{2\kappa a} = 1$, so $2\kappa a = \ln(1) = 0$, so $\kappa = 0$, and again $\psi = 0$. In all cases, then, the boundary conditions force $\psi = 0$, which is unacceptable (non-normalizable).

Problem 2.4

$$\langle x\rangle = \int x|\psi|^2 dx = \frac{2}{a}\int_0^a x\sin^2\left(\frac{n\pi}{a}x\right)dx. \qquad \text{Let } y \equiv \frac{n\pi}{a}x, \text{ so } dx = \frac{a}{n\pi}dy; \quad y: 0 \to n\pi.$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y\sin^2 y\, dy = \frac{2a}{n^2\pi^2}\left[\frac{y^2}{4} - \frac{y\sin 2y}{4} - \frac{\cos 2y}{8}\right]\Bigg|_0^{n\pi}$$

$$= \frac{2a}{n^2\pi^2}\left[\frac{n^2\pi^2}{4} - \frac{\cos 2n\pi}{8} + \frac{1}{8}\right] = \boxed{\frac{a}{2}.} \quad \text{(Independent of } n.\text{)}$$

$$\langle x^2\rangle = \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{n\pi}{a}x\right)dx = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} y^2\sin^2 y\, dy$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{y^3}{6} - \left(\frac{y^2}{4} - \frac{1}{8}\right)\sin 2y - \frac{y\cos 2y}{4}\right]_0^{n\pi}$$

$$= \frac{2a^2}{(n\pi)^3}\left[\frac{(n\pi)^3}{6} - \frac{n\pi\cos(2n\pi)}{4}\right] = \boxed{a^2\left[\frac{1}{3} - \frac{1}{2(n\pi)^2}\right].}$$

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \boxed{0.} \quad (\textit{Note}: \text{Eq. 1.33 is much faster than Eq. 1.35.})$$

$$\langle p^2\rangle = \int \psi_n^*\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2\psi_n\,dx = -\hbar^2\int\psi_n^*\left(\frac{d^2\psi_n}{dx^2}\right)dx$$
$$= (-\hbar^2)\left(-\frac{2mE_n}{\hbar^2}\right)\int\psi_n^*\psi_n\,dx = 2mE_n = \boxed{\left(\frac{n\pi\hbar}{a}\right)^2.}$$

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2\left(\frac{1}{3}-\frac{1}{2(n\pi)^2}-\frac{1}{4}\right) = \frac{a^2}{4}\left(\frac{1}{3}-\frac{2}{(n\pi)^2}\right); \quad \boxed{\sigma_x = \frac{a}{2}\sqrt{\frac{1}{3}-\frac{2}{(n\pi)^2}}.}$$

$$\sigma_p^2 = \langle p^2\rangle - \langle p\rangle^2 = \left(\frac{n\pi\hbar}{a}\right)^2; \quad \boxed{\sigma_p = \frac{n\pi\hbar}{a}.} \quad \therefore \sigma_x\sigma_p = \boxed{\frac{\hbar}{2}\sqrt{\frac{(n\pi)^2}{3}-2}.}$$

The product $\sigma_x\sigma_p$ is $\boxed{\text{smallest for } n=1;}$ in that case, $\sigma_x\sigma_p = \frac{\hbar}{2}\sqrt{\frac{\pi^2}{3}-2} = (1.136)\hbar/2 > \hbar/2.$ ✓

Problem 2.5

(a)

$$|\Psi|^2 = \Psi^*\Psi = |A|^2(\psi_1^*+\psi_2^*)(\psi_1+\psi_2) = |A|^2[\psi_1^*\psi_1+\psi_1^*\psi_2+\psi_2^*\psi_1+\psi_2^*\psi_2].$$

$$1 = \int|\Psi|^2dx = |A|^2\int[|\psi_1|^2+\psi_1^*\psi_2+\psi_2^*\psi_1+|\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1t/\hbar}+\psi_2 e^{-iE_2t/\hbar}\right] \quad \left(\text{but } \frac{E_n}{\hbar} = n^2\omega\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t}+\sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right)+\sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right].}$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t}+e^{3i\omega t}\right)+\sin^2\left(\frac{2\pi}{a}x\right)\right]$$
$$= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right].}$$

(c)

$$\langle x\rangle = \int x|\Psi(x,t)|^2dx$$
$$= \frac{1}{a}\int_0^a x\left[\sin^2\left(\frac{\pi}{a}x\right)+\sin^2\left(\frac{2\pi}{a}x\right)+2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right]dx$$

$$\int_0^a x\sin^2\left(\frac{\pi}{a}x\right)dx = \left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi}{a}x\right)}{4\pi/a} - \frac{\cos\left(\frac{2\pi}{a}x\right)}{8(\pi/a)^2}\right]\Bigg|_0^a = \frac{a^2}{4} = \int_0^a x\sin^2\left(\frac{2\pi}{a}x\right)dx.$$

$$\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)dx = \frac{1}{2}\int_0^a x\left[\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\cos\left(\frac{\pi}{a}x\right) + \frac{ax}{\pi}\sin\left(\frac{\pi}{a}x\right) - \frac{a^2}{9\pi^2}\cos\left(\frac{3\pi}{a}x\right) - \frac{ax}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{\pi^2}\big(\cos(\pi) - \cos(0)\big) - \frac{a^2}{9\pi^2}\big(\cos(3\pi) - \cos(0)\big)\right] = -\frac{a^2}{\pi^2}\left(1 - \frac{1}{9}\right) = -\frac{8a^2}{9\pi^2}.$$

$$\therefore \langle x\rangle = \frac{1}{a}\left[\frac{a^2}{4} + \frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos(3\omega t)\right] = \boxed{\frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t)\right].}$$

Amplitude: $\boxed{\frac{32}{9\pi^2}\left(\frac{a}{2}\right) = 0.3603(a/2);}$ *angular frequency:* $\boxed{3\omega = \frac{3\pi^2\hbar}{2ma^2}.}$

(d)

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\left(\frac{a}{2}\right)\left(-\frac{32}{9\pi^2}\right)(-3\omega)\sin(3\omega t) = \boxed{\frac{8\hbar}{3a}\sin(3\omega t).}$$

(e) You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H\rangle = \boxed{\frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

Problem 2.6

From Problem 2.5, we see that

$$\Psi(x,t) = \boxed{\tfrac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\tfrac{\pi}{a}x\right) + \sin\left(\tfrac{2\pi}{a}x\right)e^{-3i\omega t}e^{i\phi}\right];}$$

$$|\Psi(x,t)|^2 = \boxed{\tfrac{1}{a}\left[\sin^2\left(\tfrac{\pi}{a}x\right) + \sin^2\left(\tfrac{2\pi}{a}x\right) + 2\sin\left(\tfrac{\pi}{a}x\right)\sin\left(\tfrac{2\pi}{a}x\right)\cos(3\omega t - \phi)\right];}$$

and hence $\boxed{\langle x\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t - \phi)\right].}$ This amounts physically to starting the clock at a different time (i.e., shifting the $t = 0$ point).

If $\phi = \dfrac{\pi}{2}$, so $\Psi(x,0) = A[\psi_1(x) + i\psi_2(x)]$, then $\cos(3\omega t - \phi) = \sin(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}$.

If $\phi = \pi$, so $\Psi(x,0) = A[\psi_1(x) - \psi_2(x)]$, then $\cos(3\omega t - \phi) = -\cos(3\omega t)$; $\langle x\rangle$ starts at $\dfrac{a}{2}\left(1 + \dfrac{32}{9\pi^2}\right)$.