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Preliminaries

Elementary Logic and Set Theory

1.1 Sets and Preliminary Notations, Number Sets

Exercises

Exercise 1.1.1 If $\mathcal{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ denotes the set of all integers and $\mathcal{N} = \{1, 2, 3, \dots\}$ the set of all natural numbers, exhibit the following sets in the form $A = \{a, b, c, \dots\}$:

- (i) $\{x \in \mathcal{Z} : x^2 - 2x + 1 = 0\}$
 - (ii) $\{x \in \mathcal{Z} : 4 \leq x \leq 10\}$
 - (iii) $\{x \in \mathcal{N} : x^2 < 10\}$

 - (i) $\{1\}$
 - (ii) $\{4, 5, 6, 7, 8, 9, 10\}$
 - (iii) $\{1, 2, 3\}$
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1.2 Level One Logic

Exercises

Exercise 1.2.1 Construct the truth table for *De Morgan's Law*:

$$\sim (p \wedge q) \Leftrightarrow ((\sim p) \vee (\sim q))$$

1.3 Algebra of Sets

Exercises

Exercise 1.3.1 Of 100 students polled at a certain university, 40 were enrolled in an engineering course, 50 in a mathematics course, and 64 in a physics course. Of these, only 3 were enrolled in all three subjects, 10 were enrolled only in mathematics and engineering, 35 were enrolled only in physics and mathematics, and 18 were enrolled only in engineering and physics.

- (i) How many students were enrolled only in mathematics?
- (ii) How many of the students were not enrolled in any of these three subjects?

Let A, B, C denote the subsets of students enrolled in mathematics, the engineering course and physics, respectively. Sets: $A \cap B \cap C, A \cap B - (A \cap B \cap C), A \cap C - (A \cap B \cap C)$ and $A - (B \cup C)$ are pairwise disjoint (no two sets have a nonempty common part) and their union equals set A , see Fig. 1.1. Consequently,

$$\begin{aligned} \#(A - (B \cup C)) &= \#A - \#A \cap B \cap C - \#(A \cap B - (A \cap B \cap C)) - \#(A \cap C - (A \cap B \cap C)) \\ &= 50 - 3 - 10 - 35 = 2 \end{aligned}$$

In the same way we compute,

$$\#(B - (A \cup C)) = 9 \quad \text{and} \quad \#(C - (A \cup B)) = 8$$

Thus, the total number of students enrolled is

$$\begin{aligned} &\#(A - (B \cup C)) + \#(B - (A \cup C)) + \#(C - (A \cup B)) \\ &+ \#(A \cap B - C) + \#(A \cap C - B) + \#(B \cap C - A) \\ &+ \#(A \cap B \cap C) \\ &= 2 + 9 + 8 + 10 + 35 + 18 + 3 = 85 \end{aligned}$$

Consequently, 15 students did not enroll in any of the three classes.

Exercise 1.3.2 List all of the subsets of $A = \{1, 2, 3, 4\}$. Note: A and \emptyset are considered to be subsets of A .

$$\begin{aligned} &\emptyset, \\ &\{1\}, \\ &\{2\}, \{1, 2\}, \\ &\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \\ &\{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\} \end{aligned}$$

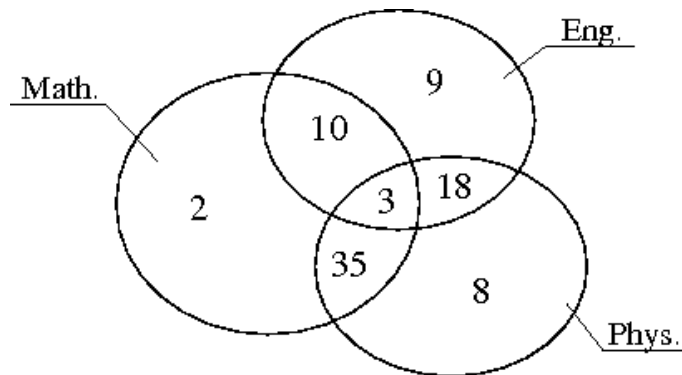
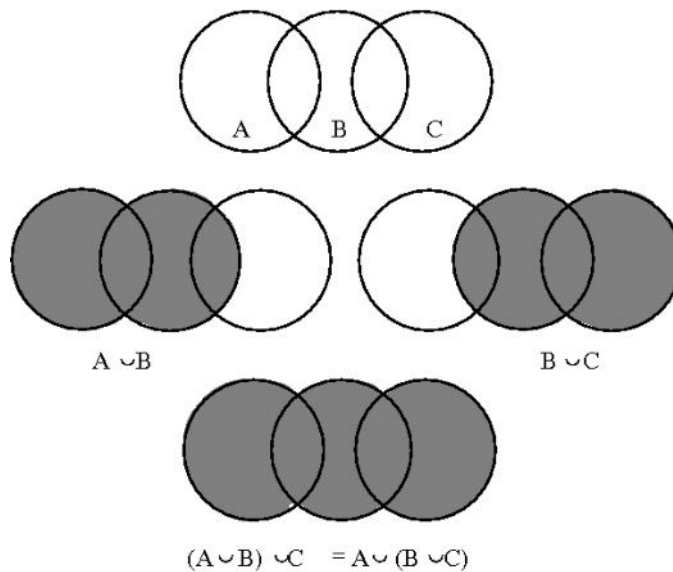
**Figure 1.1**

Illustration of Exercise 1.3.1.

Exercise 1.3.3 Construct Venn diagrams to illustrate the idempotent, commutative, associative, distributive, and identity laws. Note: some of these are trivially illustrated.

This is a very simple exercise. For example, Fig. 1.2 illustrates the associative law for the union of sets.

**Figure 1.2**

Venn diagrams illustrating the associative law for the union of sets.

Exercise 1.3.4 Construct Venn diagrams to illustrate De Morgan's Laws.

Follow Exercise 1.3.3.

Exercise 1.3.5 Prove the distributive laws.

$$\begin{array}{ll}
x \in A \cap (B \cup C) & \\
\Downarrow & \text{definition of intersection of sets} \\
x \in A \text{ and } x \in (B \cup C) & \\
\Downarrow & \text{definition of union of sets} \\
x \in A \text{ and } (x \in B \text{ or } x \in C) & \\
\Downarrow & \text{tautology: } p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r) \\
(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) & \\
\Downarrow & \text{definition of intersection of sets} \\
x \in (A \cap B) \text{ or } x \in (A \cap C) & \\
\Downarrow & \text{definition of union of sets} \\
x \in (A \cap B) \cup (A \cap C) &
\end{array}$$

In the same way,

$$\begin{array}{ll}
x \in A \cup (B \cap C) & \\
\Downarrow & \text{definition of union of sets} \\
x \in A \text{ or } x \in (B \cap C) & \\
\Downarrow & \text{definition of intersection of sets} \\
x \in A \text{ or } (x \in B \text{ and } x \in C) & \\
\Downarrow & \text{tautology: } p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \\
(x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) & \\
\Downarrow & \text{definition of union of sets} \\
x \in (A \cup B) \text{ and } x \in (A \cup C) & \\
\Downarrow & \text{definition of intersection of sets} \\
x \in (A \cup B) \cap (A \cup C) &
\end{array}$$

Exercise 1.3.6 Prove the identity laws.

In each case, one first has to identify and prove the corresponding logical law. For instance, using the truth tables, we first verify that, if f denotes a false statement, then

$$p \vee f \Leftrightarrow p$$

for an arbitrary statement p . This tautology then provides the basis for the corresponding identity law in algebra of sets:

$$A \cup \emptyset = A$$

Indeed,

$$\begin{array}{ll}
x \in (A \cup \emptyset) & \\
\Downarrow & \text{definition of union of sets} \\
x \in A \text{ or } x \in \emptyset & \\
\Downarrow & \text{tautology above} \\
x \in A &
\end{array}$$

The remaining three proofs are analogous.

Exercise 1.3.7 Prove the second of De Morgan's Laws.

$$\begin{array}{ll}
x \in A - (B \cap C) & \\
\Downarrow & \text{definition of difference of sets} \\
x \in A \text{ and } x \notin (B \cap C) & \\
\Downarrow & x \notin D \Leftrightarrow \sim (x \in D) \\
x \in A \text{ and } \sim (x \in B \cap C) & \\
\Downarrow & \text{definition of intersection} \\
x \in A \text{ and } \sim (x \in B \wedge x \in C) & \\
\Downarrow & \text{tautology: } p \wedge \sim (q \wedge r) \Leftrightarrow (p \wedge \sim q) \vee (p \wedge \sim r) \\
(x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) & \\
\Downarrow & \text{definition of difference of sets} \\
x \in (A - B) \text{ or } x \in (A - C) & \\
\Downarrow & \text{definition of union} \\
x \in (A - B) \cup (A - C) &
\end{array}$$

Exercise 1.3.8 Prove that $(A - B) \cap B = \emptyset$.

Empty set is a subset of any set, so the inclusion $\emptyset \subset (A - B) \cap B$ is obviously satisfied. To prove the converse, notice that the statement $x \in \emptyset$ is equivalent to the statement that x does not exist. Suppose now, to the contrary, that there exists an x such that $x \in (A - B) \cap B$. Then

$$\begin{array}{ll}
x \in (A - B) \cap B & \\
\Downarrow & \text{definition of intersection} \\
x \in (A - B) \text{ and } x \in B & \\
\Downarrow & \text{definition of difference} \\
(x \in A \text{ and } x \notin B) \text{ and } x \in B & \\
\Downarrow & \text{associative law for conjunction} \\
x \in A \text{ and } (x \notin B \text{ and } x \in B) & \\
\Downarrow & p \wedge \sim p \text{ is false} \\
x \in A \text{ and } x \in \emptyset & \\
\Downarrow & \text{identity law for conjunction} \\
x \in \emptyset &
\end{array}$$

In fact, the statements above are equivalent.

Exercise 1.3.9 Prove that $B - A = B \cap A'$.

$$\begin{array}{ll}
x \in B - A & \\
\Downarrow & \text{definition of difference} \\
x \in B \text{ and } x \notin A & \\
\Downarrow & \text{definition of complement} \\
x \in B \text{ and } x \in A' & \\
\Downarrow & \text{definition of intersection} \\
x \in B \cap A' &
\end{array}$$

1.4 Level Two Logic

Exercises

Exercise 1.4.1 Use Mathematical Induction to derive and prove a formula for the sum of squares of the first n positive integers:

$$\sum_{i=1}^n i^2 = 1 + 2^2 + \dots + n^2$$

This is an “inverse engineering” problem. Based on elementary integration formulas for polynomials, we expect the formula to take the form:

$$\sum_{i=1}^n i^2 = \frac{\alpha n^3 + \beta n^2 + \gamma n + \delta}{A}$$

In the proof by induction, we will need to show that:

$$\sum_{i=1}^n i^2 + (n+1)^2 = \sum_{i=1}^{n+1} i^2$$

This leads to the identity:

$$\frac{\alpha n^3 + \beta n^2 + \gamma n + \delta}{A} + (n+1)^3 = \frac{\alpha(n+1)^3 + \beta(n+1)^2 + \gamma(n+1) + \delta}{A}$$

Comparing coefficients in front of n^3 , n^2 , n , 1 on both sides, we get relations:

$$A = 3\alpha, \quad 2A = 3\alpha + 2\beta, \quad A = \alpha + \beta + \gamma$$

This leads to $\alpha = A/3$, $\beta = A/2$, $\gamma = A/6$. Choosing $A = 6$, we get $\alpha = 2$, $\beta = 3$, $\gamma = 1$. Validity of the formula for $n = 1$ implies that $\delta = 0$.

Exercise 1.4.2 Use mathematical induction to prove that the power set of a set U with n elements has 2^n elements:

$$\#U = n \quad \Rightarrow \quad \#\mathcal{P}(U) = 2^n$$

The hash symbol $\#$ replaces the phrase “number of elements of.”

- $T(1)$. Let $U = \{a\}$. Then $\mathcal{P}(U) = \{\emptyset, \{a\}\}$, so $\#\mathcal{P}(U) = 2$.
- $T(n) \Rightarrow T(n+1)$. Assume the statement has been proved for every set with n elements. Let $\#U = n+1$. Pick an arbitrary element a from set U . The power set of U can then be split into two families: subsets that do not contain element a and subsets that do contain a :

$$\mathcal{P}(U) = \mathcal{A} \cup \mathcal{B} \text{ where } \mathcal{A} = \{A \subset U : a \notin A\}, \quad \mathcal{B} = \{B \subset U : a \in B\}$$

The two families are disjoint, so $\#\mathcal{P}(U) = \#\mathcal{A} + \#\mathcal{B}$. But $\mathcal{A} = \mathcal{P}(U - \{a\})$ so, by the assumption of mathematical induction, set \mathcal{A} has 2^n elements. On the other side:

$$B \in \mathcal{B} \Leftrightarrow B - \{a\} \in \mathcal{P}(U - \{a\})$$

so family \mathcal{B} has also exactly 2^n elements. Consequently, power set $\mathcal{P}(U)$ has $2^n + 2^n = 2^{n+1}$ elements, and the proof is finished.

Another way to see the result is to recall Newton's formula:

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

In the particular case of $a = b = 1$, Newton's formula reduces to the identity:

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

Recall that Newton's symbol

$$\binom{n}{k}$$

represents the number of k -combinations of n elements, i.e., the number of different subsets with k elements from a set with n elements. As all subsets of a set U with n elements can be partitioned into subfamilies of subsets with k elements, $k = 0, 1, \dots, n$, the right-hand side of the identity above clearly represents the number of all possible subsets of set U . Obviously, in order to prove the formula above, we may have to use mathematical induction as well.

1.5 Infinite Unions and Intersections

Exercises

Exercise 1.5.1 Let $B(a, r)$ denote an open ball centered at a with radius r :

$$B(a, r) = \{x : d(x, a) < r\}$$

Here a, x are points in the Euclidean space and $d(x, a)$ denotes the (Euclidean) distance between the points. Similarly, let $\overline{B}(a, r)$ denote a closed ball centered at a with radius r :

$$\overline{B}(a, r) = \{x : d(x, a) \leq r\}$$

Notice that the open ball does not include the points on the sphere with radius r , whereas the closed ball does.