

Solution and Answer Guide

ZILL, DIFFERENTIAL EQUATIONS WITH MODELING APPLICATIONS 2024,
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END OF SECTION SOLUTIONS

EXERCISES 1.1

1. Second order; linear
2. Third order; nonlinear because of $(dy/dx)^4$
3. Fourth order; linear
4. Second order; nonlinear because of $\cos(r + u)$
5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1 + (dy/dx)^2}$
6. Second order; nonlinear because of \mathbb{R}^2
7. Third order; linear
8. Second order; nonlinear because of \dot{x}^2
9. First order; nonlinear because of $\sin(dy/dx)$
10. First order; linear
11. Writing the differential equation in the form $x(dy/dx) + y^2 = 1$, we see that it is nonlinear in y because of y^2 . However, writing it in the form $(y^2 - 1)(dx/dy) + x = 0$, we see that it is linear in x .
12. Writing the differential equation in the form $u(dv/du) + (1 + u)v = ue^u$ we see that it is linear in v . However, writing it in the form $(v + uv - ue^u)(du/dv) + u = 0$, we see that it is nonlinear in u .
13. From $y = e^{-x/2}$ we obtain $y' = -\frac{1}{2}e^{-x/2}$. Then $2y' + y = -e^{-x/2} + e^{-x/2} = 0$.

14. From $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ we obtain $dy/dt = 24e^{-20t}$, so that

$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24.$$

15. From $y = e^{3x} \cos 2x$ we obtain $y' = 3e^{3x} \cos 2x - 2e^{3x} \sin 2x$ and $y'' = 5e^{3x} \cos 2x - 12e^{3x} \sin 2x$, so that $y'' - 6y' + 13y = 0$.
16. From $y = -\cos x \ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x \ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x \ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.
17. The domain of the function, found by solving $x + 2 \geq 0$, is $[-2, \infty)$. From $y' = 1 + 2(x + 2)^{-1/2}$ we have

$$\begin{aligned}(y - x)y' &= (y - x)[1 + 2(x + 2)^{-1/2}] \\ &= y - x + 2(y - x)(x + 2)^{-1/2} \\ &= y - x + 2[x + 4(x + 2)^{1/2} - x](x + 2)^{-1/2} \\ &= y - x + 8(x + 2)^{1/2}(x + 2)^{-1/2} = y - x + 8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x = -2$.

18. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5 \tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25 \sec^2 5x$ we have

$$y' = 25(1 + \tan^2 5x) = 25 + 25 \tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

19. The domain of the function is $\{x \mid 4 - x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ or } x \neq 2\}$. From $y' = 2x/(4 - x^2)^2$ we have

$$y' = 2x \left(\frac{1}{4 - x^2}\right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

20. The function is $y = 1/\sqrt{1 - \sin x}$, whose domain is obtained from $1 - \sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1 - \sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1 - \sin x)^{-3/2} \cos x = [(1 - \sin x)^{-1/2}]^3 \cos x = y^3 \cos x.$$

An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another one is $(5\pi/2, 9\pi/2)$, and so on.

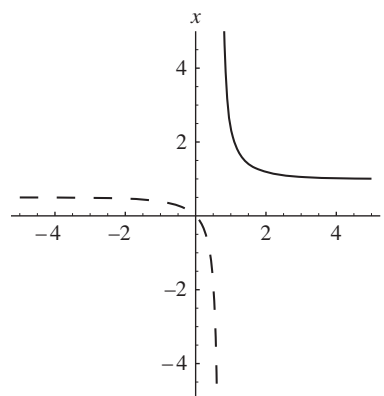
- 21.** Writing $\ln(2X - 1) - \ln(X - 1) = t$ and differentiating implicitly we obtain

$$\frac{2}{2X - 1} \frac{dX}{dt} - \frac{1}{X - 1} \frac{dX}{dt} = 1$$

$$\left(\frac{2}{2X - 1} - \frac{1}{X - 1} \right) \frac{dX}{dt} = 1$$

$$\frac{2X - 2 - 2X + 1}{(2X - 1)(X - 1)} \frac{dX}{dt} = 1$$

$$\frac{dX}{dt} = -(2X - 1)(X - 1) = (X - 1)(1 - 2X).$$



Exponentiating both sides of the implicit solution we obtain

$$\frac{2X - 1}{X - 1} = e^t$$

$$2X - 1 = Xe^t - e^t$$

$$(e^t - 1) = (e^t - 2)X$$

$$X = \frac{e^t - 1}{e^t - 2}.$$

Solving $e^t - 2 = 0$ we get $t = \ln 2$. Thus, the solution is defined on $(-\infty, \ln 2)$ or on $(\ln 2, \infty)$. The graph of the solution defined on $(-\infty, \ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2, \infty)$ is solid.

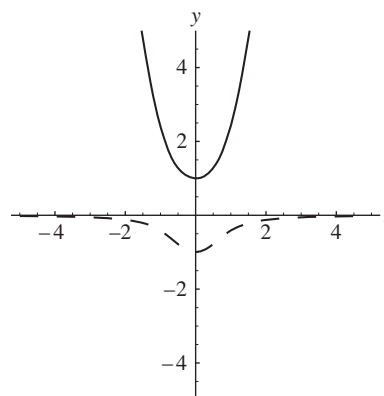
- 22.** Implicitly differentiating the solution, we obtain

$$-2x^2 \frac{dy}{dx} - 4xy + 2y \frac{dy}{dx} = 0$$

$$-x^2 dy - 2xy dx + y dy = 0$$

$$2xy dx + (x^2 - y)dy = 0.$$

Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y , we get $y = (2x^2 \pm \sqrt{4x^4 + 4})/2 = x^2 \pm \sqrt{x^4 + 1}$. Thus, two explicit solutions are $y_1 = x^2 + \sqrt{x^4 + 1}$ and $y_2 = x^2 - \sqrt{x^4 + 1}$. Both solutions are defined on $(-\infty, \infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.



23. Differentiating $P = c_1 e^t / (1 + c_1 e^t)$ we obtain

$$\begin{aligned} \frac{dP}{dt} &= \frac{(1 + c_1 e^t) c_1 e^t - c_1 e^t \cdot c_1 e^t}{(1 + c_1 e^t)^2} = \frac{c_1 e^t}{1 + c_1 e^t} \frac{[(1 + c_1 e^t) - c_1 e^t]}{1 + c_1 e^t} \\ &= \frac{c_1 e^t}{1 + c_1 e^t} \left[1 - \frac{c_1 e^t}{1 + c_1 e^t} \right] = P(1 - P). \end{aligned}$$

24. Differentiating $y = 2x^2 - 1 + c_1 e^{-2x^2}$ we obtain $\frac{dy}{dx} = 4x - 4x c_1 e^{-2x^2}$, so that

$$\frac{dy}{dx} + 4xy = 4x - 4x c_1 e^{-2x^2} + 8x^3 - 4x + 4c_1 x e^{-x^2} = 8x^3$$

25. From $y = c_1 e^{2x} + c_2 x e^{2x}$ we obtain $\frac{dy}{dx} = (2c_1 + c_2) e^{2x} + 2c_2 x e^{2x}$ and $\frac{d^2 y}{dx^2} = (4c_1 + 4c_2) e^{2x} + 4c_2 x e^{2x}$, so that

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = (4c_1 + 4c_2 - 8c_1 - 4c_2 + 4c_1) e^{2x} + (4c_2 - 8c_2 + 4c_2) x e^{2x} = 0.$$

26. From $y = c_1 x^{-1} + c_2 x + c_3 x \ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1 x^{-2} + c_2 + c_3 + c_3 \ln x + 8x,$$

$$\frac{d^2 y}{dx^2} = 2c_1 x^{-3} + c_3 x^{-1} + 8,$$

and

$$\frac{d^3 y}{dx^3} = -6c_1 x^{-4} - c_3 x^{-2},$$

so that

$$\begin{aligned} x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1) x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2) x \\ &\quad + (-c_3 + c_3) x \ln x + (16 - 8 + 4) x^2 = 12x^2 \end{aligned}$$

In Problems 25–28, we use the Product Rule and the derivative of an integral ((12) of this section):

$$\frac{d}{dx} \int_a^x g(t) dt = g(x).$$

27. Differentiating $y = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt$ we obtain $\frac{dy}{dx} = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{e^{-3x}}{x} \cdot e^{3x}$ or

$$\frac{dy}{dx} = e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{1}{x}, \text{ so that}$$

$$\begin{aligned} x \frac{dy}{dx} - 3xy &= x \left(e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + \frac{1}{x} \right) - 3x \left(e^{3x} \int_1^x \frac{e^{-3t}}{t} dt \right) \\ &= x e^{3x} \int_1^x \frac{e^{-3t}}{t} dt + 1 - 3x e^{3x} \int_1^x \frac{e^{-3t}}{t} dt = 1 \end{aligned}$$

28. Differentiating $y = \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt$ we obtain $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \frac{\cos x}{\sqrt{x}} \cdot \sqrt{x}$ or

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \cos x, \text{ so that}$$

$$\begin{aligned} 2x \frac{dy}{dx} - y &= 2x \left(\frac{1}{2\sqrt{x}} \int_4^x \frac{\cos t}{\sqrt{t}} dt + \cos x \right) - \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt \\ &= \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt + 2x \cos x - \sqrt{x} \int_4^x \frac{\cos t}{\sqrt{t}} dt = 2x \cos x \end{aligned}$$

29. Differentiating $y = \frac{5}{x} + \frac{10}{x} \int_1^x \frac{\sin t}{t} dt$ we obtain $\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{\sin x}{x} \cdot \frac{10}{x}$ or

$$\frac{dy}{dx} = -\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{10 \sin x}{x^2}, \text{ so that}$$

$$\begin{aligned} x^2 \frac{dy}{dx} + xy &= x^2 \left(-\frac{5}{x^2} - \frac{10}{x^2} \int_1^x \frac{\sin t}{t} dt + \frac{10 \sin x}{x^2} \right) + x \left(\frac{5}{x} + \frac{10}{x} \int_1^x \frac{\sin t}{t} dt \right) \\ &= -5 - 10 \int_1^x \frac{\sin t}{t} dt + 10 \sin x + 5 + 10 \int_1^x \frac{\sin t}{t} dt = 10 \sin x \end{aligned}$$

30. Differentiating $y = e^{-x^2} + e^{-x^2} \int_0^x e^{t^2} dt$ we obtain $\frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + e^{x^2} \cdot e^{-x^2}$.

$$\text{or } \frac{dy}{dx} = -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1, \text{ so that}$$

$$\begin{aligned} \frac{dy}{dx} + 2xy &= \left(-2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1 \right) + 2x \left(e^{-x^2} + e^{-x^2} \int_0^x e^{t^2} dt \right) \\ &= -2xe^{-x^2} - 2xe^{-x^2} \int_0^x e^{t^2} dt + 1 + 2xe^{-x^2} + 2xe^{-x^2} \int_0^x e^{t^2} dt = 1 \end{aligned}$$

31. From

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

we obtain

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0 \end{cases}$$

so that $xy' - 2y = 0$.

32. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x \rightarrow 0^-} y(x) = 5$ and $\lim_{x \rightarrow 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

33. Force the function $y = e^{mx}$ into the equation $y' + 2y = 0$ to get

$$(e^{mx})' + 2(e^{mx}) = 0$$

$$me^{mx} + 2e^{mx} = 0$$

$$e^{mx}(m + 2) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -2$ and so $y = e^{-2x}$ is a solution.

34. Force the function $y = e^{mx}$ into the equation $3y' - 4y = 0$ to get

$$3(e^{mx})' - 4(e^{mx}) = 0$$

$$3me^{mx} - 4e^{mx} = 0$$

$$e^{mx}(3m - 4) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = 4/3$ and so $y = e^{4x/3}$ is a solution.

35. Force the function $y = e^{mx}$ into the equation $y'' - 5y' + 6y = 0$ to get

$$(e^{mx})'' - 5(e^{mx})' + 6(e^{mx}) = 0$$

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = 0$$

$$e^{mx}(m^2 - 5m + 6) = 0$$

$$e^{mx}(m - 2)(m - 3) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = 2$ and $m = 3$ therefore $y = e^{2x}$ and $y = e^{3x}$ are solutions.

36. Force the function $y = e^{mx}$ into the equation $2y'' + 9y' - 5y = 0$ to get

$$2(e^{mx})'' + 9(e^{mx})' - 5(e^{mx}) = 0$$

$$2m^2e^{mx} + 9me^{mx} - 5e^{mx} = 0$$

$$e^{mx}(2m^2 + 9m - 5) = 0$$

$$e^{mx}(m + 5)(2m - 1) = 0$$

Now since $e^{mx} > 0$ for all values of x , we must have $m = -5$ and $m = 1/2$ therefore $y = e^{-5x}$ and $y = e^{x/2}$ are solutions.

37. Force the function $y = x^m$ into the equation $xy'' + 2y' = 0$ to get

$$\begin{aligned}x \cdot (x^m)'' + 2(x^m)' &= 0 \\x \cdot m(m-1)x^{m-2} + 2mx^{m-1} &= 0 \\(m^2 - m)x^{m-1} + 2mx^{m-1} &= 0 \\x^{m-1}[m^2 + m] &= 0 \\x^{m-1}[m(m+1)] &= 0\end{aligned}$$

The last line implies that $m = 0$ and $m = -1$ therefore $y = x^0 = 1$ and $y = x^{-1}$ are solutions.

38. Force the function $y = x^m$ into the equation $4x^2y'' + y = 0$ to get

$$\begin{aligned}4x^2(x^m)'' + (x^m) &= 0 \\4x^2 \cdot m(m-1)x^{m-2} + x^m &= 0 \\4(m^2 - m)x^m + x^m &= 0 \\x^m[4m^2 - 4m + 1] &= 0 \\x^m[(2m-1)^2] &= 0\end{aligned}$$

The last line implies that $m = 1/2$ therefore $y = x^{1/2} = \sqrt{x}$ is a solutions.

39. Force the function $y = x^m$ into the equation $x^2y'' - 7xy' + 15y = 0$ to get

$$\begin{aligned}x^2 \cdot (x^m)'' - 7x \cdot (x^m)' + 15(x^m) &= 0 \\x^2 \cdot m(m-1)x^{m-2} - 7x \cdot mx^{m-1} + 15x^m &= 0 \\(m^2 - m)x^m - 7mx^m + 15x^m &= 0 \\x^m[m^2 - 8m + 15] &= 0 \\x^m[(m-3)(m-5)] &= 0\end{aligned}$$

The last line implies that $m = 3$ and $m = 5$ therefore $y = x^3$ and $y = x^5$ are solutions.

40. Force the function $y = x^m$ into the equation $x^2y''' - 3xy'' + 3y' = 0$ to get

$$\begin{aligned} x^2 \cdot (x^m)''' - 3x \cdot (x^m)'' + 3(x^m)' &= 0 \\ x^2 \cdot m(m-1)(m-2)x^{m-3} - 3x \cdot m(m-1)x^{m-2} + 3 \cdot mx^{m-1} &= 0 \\ (m^3 - 3m^2 + 2m)x^{m-1} - 3(m^2 - m)x^{m-1} + 3mx^{m-1} &= 0 \\ x^{m-1}[m^3 - 6m^2 + 8m] &= 0 \\ x^m[m(m-2)(m-4)] &= 0 \end{aligned}$$

The last line implies that $m = 0$, $m = 2$, and $m = 4$ therefore $y = x^0 = 1$, $y = x^2$, and $y = x^4$ are solutions.

In Problems 41–44, we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$

- 41.** Solving $5c = 10$ we see that $y = 2$ is a constant solution.
42. Solving $c^2 + 2c - 3 = (c + 3)(c - 1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.
43. Since $1/(c - 1) = 0$ has no solutions, the differential equation has no constant solutions.
44. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.
45. Substituting $y = (x + c_1)^2$ into differential equation yields,

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= 4y \\ [2(x + c_1)]^2 &= 4(x + c_1)^2 \\ 4(x + c_1)^2 &= 4(x + c_1)^2 \end{aligned}$$

Both sides of the differential equation are zero when $y = 0$. No value of c_1 in the family of solutions gives $y = 0$ and thus the trivial solution $y = 0$ is a singular solution.

46. Substituting $y = 3 \sin(x + c_1)$ into differential equation yields,

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= 9 - y^2 \\ [3 \cos(x + c_1)]^2 &= 9 - (3 \sin(x + c_1))^2 \\ 9 \cos^2(x + c_1) &= 9 - 9 \sin^2(x + c_1) \\ 9 \cos^2(x + c_1) &= 9 - 9[1 - \cos^2(x + c_1)] \\ 9 \cos^2(x + c_1) &= 9 - 9 + 9 \cos^2(x + c_1) \\ 9 \cos^2(x + c_1) &= 9 \cos^2(x + c_1) \end{aligned}$$

Both sides of the differential equation are zero when $y = 3$. No value of c_1 in the family of solutions gives $y = 3$ and thus the solution $y = 3$ is a singular solution.

- 47.** Rewriting the family of solutions, we get $\sqrt{16 - y^2} = x - c_1$ and $16 - y^2 = (x - c_1)^2$. Using implicit differentiation results in $y \frac{dy}{dx} = c_1 - x$. Substituting $y \frac{dy}{dx}$ and $\sqrt{16 - y^2}$ into the differential equation yields,

$$y \frac{dy}{dx} + \sqrt{16 - y^2} = 0$$

$$(c_1 - x) + (x - c_1) = 0$$

Both sides of the differential equation are zero when $y = \pm 4$. No value of c_1 in the family of solutions gives $y = \pm 4$ and thus the solutions $y = \pm 4$ are singular solutions.

- 48.** Substituting $y = x - (x - c_1)^2$ into differential equation yields,

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 4y = 4x - 1$$

$$(1 - 2(x - c_1))^2 - 2(1 - 2(x - c_1)) + 4(x - (x - c_1)^2) = 4x - 1$$

$$(1 + 4x^2 - 8xc - 4x + 4c + 4c^2) + (-2 + 4x - 4c_1) + (4x - 4x^2 + 8xc_1 - 4c_1^2) = 4x - 1$$

$$4x - 1 = 4x - 1$$

Both sides of the differential equation are $4x - 1$ when $y = x$. No value of c_1 in the family of solutions gives $y = x$ and thus the solution $y = x$ is a singular solution.

- 49.** From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}.$$

- 50.** From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \quad \text{and} \quad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}.$$