Chapter 1

Introduction to Probability

1.2 Interpretations of Probability

Commentary

It is interesting to have the students determine some of their own subjective probabilities. For example, let X denote the temperature at noon tomorrow outside the building in which the class is being held. Have each student determine a number x_1 such that the student considers the following two possible outcomes to be equally likely: $X \le x_1$ and $X > x_1$. Also, have each student determine numbers x_2 and x_3 (with $x_2 < x_3$) such that the student considers the following three possible outcomes to be equally likely: $X \le x_2$, $x_2 < X < x_3$, and $X \ge x_3$. Determinations of more than three outcomes that are considered to be equally likely can also be made. The different values of x_1 determined by different members of the class should be discussed, and also the possibility of getting the class to agree on a common value of x_1.

Similar determinations of equally likely outcomes can be made by the students in the class for quantities such as the following ones which were found in the 1973 World Almanac and Book of Facts: the number of freight cars that were in use by American railways in 1960 (1,690,396), the number of banks in the United States which closed temporarily or permanently in 1931 on account of financial difficulties (2,294), and the total number of telephones which were in service in South America in 1971 (6,137,000).

1.4 Set Theory

Solutions to Exercises

1. Assume that $x \in B^c$. We need to show that $x \in A^c$. We shall show this indirectly. Assume, to the contrary, that $x \in A$. Then $x \in B$ because $A \subset B$. This contradicts $x \in B^c$. Hence $x \in A$ is false and $x \in A^c$.

2. First, show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. That is, $x \in A$ and either $x \in B$ or $x \in C$ (or both). So either ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) or both. That is, either $x \in A \cap B$ or $x \in A \cap C$. This is what it means to say that $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Basically, running these steps backwards shows that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

3. To prove the first result, let $x \in (A \cup B)^c$. This means that x is not in $A \cup B$. In other words, x is neither in A nor in B. Hence $x \in A^c$ and $x \in B^c$. So $x \in A^c \cap B^c$. This proves that $(A \cup B)^c \subset A^c \cap B^c$. Next, suppose that $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$. So x is neither in A nor in B, so it can't be in $A \cup B$. Hence $x \in (A \cup B)^c$. This shows that $A^c \cap B^c \subset (A \cup B)^c$. The second result follows from the first by applying the first result to A^c and B^c and then taking complements of both sides.

4. To see that $A \cap B$ and $A \cap B^c$ are disjoint, let $x \in A \cap B$. Then $x \in B$, hence $x \notin B^c$ and so $x \notin A \cap B^c$. So no element of $A \cap B$ is in $A \cap B^c$, hence the two events are disjoint. To prove that $A = (A \cap B) \cup (A \cap B^c)$, we shall show that each side is a subset of the other side. First, let $x \in A$. Either $x \in B$ or $x \in B^c$. If $x \in B$, then $x \in A \cap B$. If $x \in B^c$, then $x \in A \cap B^c$. Either way, $x \in (A \cap B) \cup (A \cap B^c)$. So every element of A is an element of $(A \cap B) \cup (A \cap B^c)$ and we conclude that $A \subset (A \cap B) \cup (A \cap B^c)$. Finally, let $x \in (A \cap B) \cup (A \cap B^c)$. Then either $x \in A \cap B$, in which case $x \in A$, or $x \in A \cap B^c$, in which case $x \in A$. Either way $x \in A$, so every element of $(A \cap B) \cup (A \cap B^c)$ is also an element of A and $(A \cap B) \cup (A \cap B^c) \subset A$.

5. To prove the first result, let $x \in (\cup_i A_i)^c$. This means that x is not in $\cup_i A_i$. In other words, for every $i \in I$, x is not in A_i. Hence for every $i \in I$, $x \in A_i^c$. So $x \in \cap_i A_i^c$. This proves that $(\cup_i A_i)^c \subset \cap_i A_i^c$. Next, suppose that $x \in \cap_i A_i^c$. Then $x \in A_i^c$ for every $i \in I$. So for every $i \in I$, x is not in A_i. So x can't be in $\cup_i A_i$. Hence $x \in (\cup_i A_i)^c$. This shows that $\cap_i A_i^c \subset (\cup_i A_i)^c$. The second result follows from the first by applying the first result to A_i^c for $i \in I$ and then taking complements of both sides.

6. (a) Blue card numbered 2 or 4.
   (b) Blue card numbered 5, 6, 7, 8, 9, or 10.
   (c) Any blue card or a red card numbered 1, 2, 3, 4, 6, 8, or 10.
   (d) Blue card numbered 2, 4, 6, 8, or 10, or red card numbered 2 or 4.
   (e) Red card numbered 5, 7, or 9.

7. (a) These are the points not in A, hence they must be either below 1 or above 5. That is $A^c = \{x : x < 1 \text{ or } x > 5\}$.
   (b) These are the points in either A or B or both. So they must be between 1 and 5 or between 3 and 7. That is, $A \cup B = \{x : 1 \leq x \leq 7\}$.
   (c) These are the points in B but not in C. That is $BC^c = \{x : 3 < x \leq 7\}$. (Note that $B \subset C^c$.)
   (d) These are the points in none of the three sets, namely $A^cB^cC^c = \{x : 0 < x < 1 \text{ or } x > 7\}$.
   (e) These are the points in the answer to part (b) and in C. There are no such values and $(A \cup B)C = \emptyset$.

8. Blood type A reacts only with anti-A, so type A blood corresponds to $A \cap B^c$. Type B blood reacts only with anti-B, so type B blood corresponds to A^cB. Type AB blood reacts with both, so $A \cap B$ characterizes type AB blood. Finally, type O reacts with neither antigen, so type O blood corresponds to the event A^cB^c.

9. (a) For each n, $B_n = B_{n+1} \cup A_n$, hence $B_n \supset B_{n+1}$ for all n. For each n, $C_{n+1} \cap A_n = C_n$, so $C_n \subset C_{n+1}$.
   (b) Suppose that $x \in \cap_{n=1}^{\infty} B_n$. Then $x \in B_n$ for all n. That is, $x \in \cup_{i=n}^{\infty} A_i$ for all n. For $n = 1$, there exists $i \geq n$ such that $x \in A_i$. Assume to the contrary that there are at most finitely many i such that $x \in A_i$. Let m be the largest such i. For $n = m + 1$, we know that there is $i \geq n$ such that $x \in A_i$. This contradicts m being the largest i such that $x \in A_i$. Hence, x is in infinitely many A_i. For the other direction, assume that x is in infinitely many A_i. Then, for every n, there is a value of $j > n$ such that $x \in A_j$, hence $x \in \cup_{i=n}^{\infty} A_i = B_n$ for every n and $x \in \cap_{n=1}^{\infty} B_n$.
   (c) Suppose that $x \in \cup_{n=1}^{\infty} C_n$. That is, there exists n such that $x \in C_n = \cap_{i=n}^{\infty} A_i$, so $x \in A_i$ for all $i \geq n$. So, there at most finitely many i (a subset of $1, \ldots, n-1$) such that $x \notin A_i$. Finally, suppose that $x \in A_i$ for all but finitely many i. Let k be the last i such that $x \notin A_i$. Then $x \in A_i$ for all $i \geq k + 1$, hence $x \in \cap_{i=k+1}^{\infty} A_i = C_{k+1}$. Hence $x \in \cup_{n=1}^{\infty} C_n$.

10. (a) All three dice show even numbers if and only if all three of A, B, and C occur. So, the event is $A \cap B \cap C$.
    (b) None of the three dice show even numbers if and only if all three of A^c, B^c, and C^c occur. So, the event is $A^c \cap B^c \cap C^c$.
    (c) At least one die shows an odd number if and only if at least one of A^c, B^c, and C^c occur. So, the event is $A^c \cup B^c \cup C^c$.
    (d) At most two dice show odd numbers if and only if at least one die shows an even number, so the event is $A \cup B \cup C$. This can also be expressed as the union of the three events of the form $A \cap B \cap C^c$ where exactly one die shows odd together with the three events of the form $A \cap B^c \cap C^c$ where exactly two dice show odd together with the even $A \cap B \cap C$ where no dice show odd.
    (e) We can enumerate all the sums that are no greater than 5: $1+1+1$, $2+1+1$, $1+2+1$, $1+1+2$, $2+2+1$, $2+1+2$, and $1+2+2$. The first of these corresponds to the event $A_1 \cap B_1 \cap C_1$, the second to $A_2 \cap B_1 \cap C_1$, etc. The union of the seven such events is what is requested, namely

$$(A_1 \cap B_1 \cap C_1) \cup (A_2 \cap B_1 \cap C_1) \cup (A_1 \cap B_2 \cap C_1) \cup (A_1 \cap B_1 \cap C_2) \cup (A_2 \cap B_2 \cap C_1) \cup (A_2 \cap B_1 \cap C_2) \cup (A_1 \cap B_2 \cap C_2).$$

11. (a) All of the events mentioned can be determined by knowing the voltages of the two subcells. Hence the following set can serve as a sample space

$$S = \{(x, y) : 0 \le x \le 5 \text{ and } 0 \le y \le 5\},$$

    where the first coordinate is the voltage of the first subcell and the second coordinate is the voltage of the second subcell. Any more complicated set from which these two voltages can be determined could serve as the sample space, so long as each outcome could at least hypothetically be learned.
    (b) The power cell is functional if and only if the sum of the voltages is at least 6. Hence, $A = \{(x, y) \in S : x + y \ge 6\}$. It is clear that $B = \{(x, y) \in S : x = y\}$ and $C = \{(x, y) \in S : x > y\}$. The powercell is not functional if and only if the sum of the voltages is less than 6. It needs less than one volt to be functional if and only if the sum of the voltages is greater than 5. The intersection of these two is the event $D = \{(x, y) \in S : 5 < x + y < 6\}$. The restriction "$\in S$" that appears in each of these descriptions guarantees that the set is a subset of S. One could leave off this restriction and add the two restrictions $0 \le x \le 5$ and $0 \le y \le 5$ to each set.
    (c) The description can be worded as "the power cell is not functional, and needs at least one more volt to be functional, and both subcells have the same voltage." This is the intersection of A^c, D^c, and B. That is, $A^c \cap D^c \cap B$. The part of D^c in which $x + y \ge 6$ is not part of this set because of the intersection with A^c.
    (d) We need the intersection of A^c (not functional) with C^c (second subcell at least as big as first) and with B^c (subcells are not the same). In particular, $C^c \cap B^c$ is the event that the second subcell is strictly higher than the first. So, the event is $A^c \cap B^c \cap C^c$.

1.5 The Definition of Probability

Solutions to Exercises

1. Define the following events:

$$\begin{aligned} A &= \{\text{the selected ball is red}\}, \\ B &= \{\text{the selected ball is white}\}, \\ C &= \{\text{the selected ball is either blue, yellow, or green}\}. \end{aligned}$$

We are asked to find $\Pr(C)$. The three events A, B, and C are disjoint and $A \cup B \cup C = S$. So $1 = \Pr(A) + \Pr(B) + \Pr(C)$. We are told that $\Pr(A) = 1/5$ and $\Pr(B) = 2/5$. It follows that $\Pr(C) = 2/5$.

2. Let B be the event that a boy is selected, and let G be the event that a girl is selected. We are told that $B \cup G = S$, so $G = B^c$. Since $\Pr(B) = 0.3$, it follows that $\Pr(G) = 0.7$.

3. (a) If A and B are disjoint then $B \subset A^c$ and $BA^c = B$, so $\Pr(BA^c) = \Pr(B) = 1/2$.
   (b) If $A \subset B$, then $B = A \cup (BA^c)$ with A and BA^c disjoint. So $\Pr(B) = \Pr(A) + \Pr(BA^c)$. That is, $1/2 = 1/3 + \Pr(BA^c)$, so $\Pr(BA^c) = 1/6$.
   (c) According to Theorem 1.4.11, $B = (BA) \cup (BA^c)$. Also, BA and BA^c are disjoint so, $\Pr(B) = \Pr(BA) + \Pr(BA^c)$. That is, $1/2 = 1/8 + \Pr(BA^c)$, so $\Pr(BA^c) = 3/8$.

4. Let E_1 be the event that student A fails and let E_2 be the event that student B fails. We want $\Pr(E_1 \cup E_2)$. We are told that $\Pr(E_1) = 0.5$, $\Pr(E_2) = 0.2$, and $\Pr(E_1E_2) = 0.1$. According to Theorem 1.5.7, $\Pr(E_1 \cup E_2) = 0.5 + 0.2 - 0.1 = 0.6$.

5. Using the same notation as in Exercise 4, we now want $\Pr(E_1^c \cap E_2^c)$. According to Theorems 1.4.9 and 1.5.3, this equals $1 - \Pr(E_1 \cup E_2) = 0.4$.

6. Using the same notation as in Exercise 4, we now want $\Pr([E_1 \cap E_2^c] \cup [E_1^c \cap E_2])$. These two events are disjoint, so

$$\Pr([E_1 \cap E_2^c] \cup [E_1^c \cap E_2]) = \Pr(E_1 \cap E_2^c) + \Pr(E_1^c \cap E_2).$$

Use the reasoning from part (c) of Exercise 3 above to conclude that

$$\begin{aligned} \Pr(E_1 \cap E_2^c) &= \Pr(E_1) - \Pr(E_1 \cap E_2) = 0.4, \\ \Pr(E_1^c \cap E_2) &= \Pr(E_2) - \Pr(E_1 \cap E_2) = 0.1. \end{aligned}$$

It follows that the probability we want is 0.5.

7. Rearranging terms in Eq. (1.5.1) of the text, we get

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.4 + 0.7 - \Pr(A \cup B) = 1.1 - \Pr(A \cup B).$$

So $\Pr(A \cap B)$ is largest when $\Pr(A \cup B)$ is smallest and vice-versa. The smallest possible value for $\Pr(A \cup B)$ occurs when one of the events is a subset of the other. In the present exercise this could only happen if $A \subset B$, in which case $\Pr(A \cup B) = \Pr(B) = 0.7$, and $\Pr(A \cap B) = 0.4$. The largest possible value of $\Pr(A \cup B)$ occurs when either A and B are disjoint or when $A \cup B = S$. The former is not possible since the probabilities are too large, but the latter is possible. In this case $\Pr(A \cup B) = 1$ and $\Pr(A \cap B) = 0.1$.

8. Let A be the event that a randomly selected family subscribes to the morning paper, and let B be the event that a randomly selected family subscribes to the afternoon paper. We are told that $\Pr(A) = 0.5$, $\Pr(B) = 0.65$, and $\Pr(A \cup B) = 0.85$. We are asked to find $\Pr(A \cap B)$. Using Theorem 1.5.7 in the text we obtain

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.5 + 0.65 - 0.85 = 0.3.$$

9. The required probability is

$$\begin{aligned}\Pr(A \cap B^C) + \Pr(A^C B) &= [\Pr(A) - \Pr(A \cap B)] + [\Pr(B) - \Pr(A \cap B)] \\ &= \Pr(A) + \Pr(B) - 2\Pr(A \cap B).\end{aligned}$$

10. Theorem 1.4.11 says that $A = (A \cap B) \cup (A \cap B^c)$. Clearly the two events $A \cap B$ and $A \cap B^c$ are disjoint. It follows from Theorem 1.5.6 that $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B^c)$.

11. (a) The set of points for which $(x-1/2)^2 + (y-1/2)^2 < 1/4$ is the interior of a circle that is contained in the unit square. (Its center is $(1/2, 1/2)$ and its radius is $1/2$.) The area of this circle is $\pi/4$, so the area of the remaining region (what we want) is $1 - \pi/4$.

    (b) We need the area of the region between the two lines $y = 1/2 - x$ and $y = 3/2 - x$. The remaining area is the union of two right triangles with base and height both equal to $1/2$. Each triangle has area $1/8$, so the region between the two lines has area $1 - 2/8 = 3/4$.

    (c) We can use calculus to do this. We want the area under the curve $y = 1 - x^2$ between $x = 0$ and $x = 1$. This equals

    $$\int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\Big|_{x=0}^1 = \frac{2}{3}.$$

    (d) The area of a line is 0, so the probability of a line segment is 0.

12. The events $B_1, B_2, \ldots$ are disjoint, because the event B_1 contains the points in A_1, the event B_2 contains the points in A_2 but not in A_1, the event B_3 contains the points in A_3 but not in A_1 or A_2, etc. By this same reasoning, it is seen that $\cup_{i=1}^n A_i = \cup_{i=1}^n B_i$ and $\cup_{i=1}^\infty A_i = \cup_{i=1}^\infty B_i$. Therefore,

$$\Pr\left(\bigcup_{i=1}^n A_i\right) = \Pr\left(\bigcup_{i=1}^n B_i\right)$$

and

$$\Pr\left(\bigcup_{i=1}^\infty A_i\right) = \Pr\left(\bigcup_{i=1}^\infty B_i\right).$$

However, since the events $B_1, B_2, \ldots$ are disjoint,

$$\Pr\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n Pr(B_i)$$

and

$$\Pr\left(\bigcup_{i=1}^\infty B_i\right) = \sum_{i=1}^\infty \Pr(B_i).$$

13. We know from Exercise 12 that

$$\Pr\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \Pr(B_i).$$

Furthermore, from the definition of the events $B_1, \dots, B_n$ it is seen that $B_i \subset A_i$ for $i = 1, \dots, n$. Therefore, by Theorem 1.5.4, $\Pr(B_i) \le \Pr(A_i)$ for $i = 1, \dots, n$. It now follows that

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \Pr(A_i).$$

(Of course, if the events $A_1, \dots, A_n$ are disjoint, there is equality in this relation.)

For the second part, apply the first part with A_i replaced by A_i^c for $i = 1, \dots, n$. We get

$$\Pr\left(\bigcup A_i^c\right) \le \sum_{i=1}^{n} \Pr(A_i^c). \tag{S.1.1}$$

Exercise 5 in Sec. 1.4 says that the left side of (S.1.1) is $\Pr\left([\bigcap A_i]^c\right)$. Theorem 1.5.3 says that this last probability is $1 - \Pr\left(\bigcap A_i\right)$. Hence, we can rewrite (S.1.1) as

$$1 - \Pr\left(\bigcap A_i\right) \le \sum_{i=1}^{n} \Pr(A_i^c).$$

Finally take one minus both sides of the above inequality (which reverses the inequality) and produces the desired result.

14. First, note that the probability of type AB blood is $1-(0.5+0.34+0.12) = 0.04$ by using Theorems 1.5.2 and 1.5.3.
    (a) The probability of blood reacting to anti-A is the probability that the blood is either type A or type AB. Since these are disjoint events, the probability is the sum of the two probabilities, namely $0.34 + 0.04 = 0.38$. Similarly, the probability of reacting with anti-B is the probability of being either type B or type AB, $0.12 + 0.04 = 0.16$.
    (b) The probability that both antigens react is the probability of type AB blood, namely 0.04.

1.6 Finite Sample Spaces

Solutions to Exercises

1. The safe way to obtain the answer at this stage of our development is to count that 18 of the 36 outcomes in the sample space yield an odd sum. Another way to solve the problem is to note that regardless of what number appears on the first die, there are three numbers on the second die that will yield an odd sum and three numbers that will yield an even sum. Either way the probability is 1/2.

2. The event whose probability we want is the complement of the event in Exercise 1, so the probability is also 1/2.

3. The only differences greater than or equal to 3 that are available are 3, 4 and 5. These large difference only occur for the six outcomes in the upper right and the six outcomes in the lower left of the array in Example 1.6.5 of the text. So the probability we want is $1 - 12/36 = 2/3$.

4. Let x be the proportion of the school in grade 3 (the same as grades 2–6). Then $2x$ is the proportion in grade 1 and $1 = 2x + 5x = 7x$. So $x = 1/7$, which is the probability that a randomly selected student will be in grade 3.

5. The probability of being in an odd-numbered grade is $2x + x + x = 4x = 4/7$.

6. Assume that all eight possible combinations of faces are equally likely. Only two of those combinations have all three faces the same, so the probability is 1/4.

7. The possible genotypes of the offspring are aa and Aa, since one parent will definitely contribute an a, while the other can contribute either A or a. Since the parent who is Aa contributes each possible allele with probability 1/2 each, the probabilities of the two possible offspring are each 1/2 as well.

8. (a) The sample space contains 12 outcomes: (Head, 1), (Tail, 1), (Head, 2), (Tail, 2), etc.
   (b) Assume that all 12 outcomes are equally likely. Three of the outcomes have Head and an odd number, so the probability is 1/4.

1.7 Counting Methods

Commentary

If you wish to stress computer evaluation of probabilities, then there are programs for computing factorials and log-factorials. For example, in the statistical software R, there are functions `factorial` and `lfactorial` that compute these. If you cover Stirling's formula (Theorem 1.7.5), you can use these functions to illustrate the closeness of the approximation.

Solutions to Exercises

1. Each pair of starting day and leap year/no leap year designation determines a calendar, and each calendar correspond to exactly one such pair. Since there are seven days and two designations, there are a total of $7 \times 2 = 14$ different calendars.

2. There are 20 ways to choose the student from the first class, and no matter which is chosen, there are 18 ways to choose the student from the second class. No matter which two students are chosen from the first two classes, there are 25 ways to choose the student from the third class. The multiplication rule can be applied to conclude that the total number of ways to choose the three members is $20 \times 18 \times 25 = 9000$.

3. This is a simple matter of permutations of five distinct items, so there are $5! = 120$ ways.

4. There are six different possible shirts, and no matter what shirt is picked, there are four different slacks. So there are 24 different combinations.

5. Let the sample space consist of all four-tuples of dice rolls. There are $6^4 = 1296$ possible outcomes. The outcomes with all four rolls different consist of all of the permutations of six items taken four at a time. There are $P_{6,4} = 360$ of these outcomes. So the probability we want is $360/1296 = 5/18$.

6. With six rolls, there are $6^6 = 46656$ possible outcomes. The outcomes with all different rolls are the permutations of six distinct items. There are $6! = 720$ outcomes in the event of interest, so the probability is $720/46656 = 0.01543$.

7. There are 20^{12} possible outcomes in the sample space. If the 12 balls are to be thrown into different boxes, the first ball can be thrown into any one of the 20 boxes, the second ball can then be thrown into any one of the other 19 boxes, etc. Thus, there are $20 \cdot 19 \cdot 18 \cdots 9$ possible outcomes in the event. So the probability is $20!/[8!20^{12}]$.

8. There are 7^5 possible outcomes in the sample space. If the five passengers are to get off at different floors, the first passenger can get off at any one of the seven floors, the second passenger can then get off at any one of the other six floors, etc. Thus, the probability is

$$\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{7^5} = \frac{360}{2401}.$$

9. There are 6! possible arrangements in which the six runners can finish the race. If the three runners from team A finish in the first three positions, there are 3! arrangements of these three runners among these three positions and there are also 3! arrangements of the three runners from team B among the last three positions. Therefore, there are 3! × 3! arrangements in which the runners from team A finish in the first three positions and the runners from team B finish in the last three positions. Thus, the probability is $(3!3!)/6! = 1/20$.

10. We can imagine that the 100 balls are randomly ordered in a list, and then drawn in that order. Thus, the required probability in part (a), (b), or (c) of this exercise is simply the probability that the first, fiftieth, or last ball in the list is red. Each of these probabilities is the same $\frac{r}{100}$, because of the random order of the list.

11. In terms of factorials, $P_{n,k} = n!/[k!(n-k)!]$. Since we are assuming that n and $n = k$ are large, we can use Stirling's formula to approximate both of them. The approximation to $n!$ is $(2\pi)^{1/2} n^{n+1/2} e^{-n}$, and the approximation to $(n-k)!$ is $(2\pi)^{1/2}(n-k)^{n-k+1/2} e^{-n+k}$. The approximation to the ratio is the ratio of the approximations because the ratio of each approximation to its corresponding factorial converges to 1. That is,

$$\frac{n!}{k!(n-k)!} \approx \frac{(2\pi)^{1/2} n^{n+1/2} e^{-n}}{k!(2\pi)^{1/2}(n-k)^{n-k+1/2} e^{-n+k}} = \frac{e^{-k} n^k}{k!} \left(1 - \frac{k}{n}\right)^{-n-k-1/2}.$$

Further simplification is available if one assumes that k is small compared to n, that is $k/n \approx 0$. In this case, the last factor is approximately e^k, and the whole approximation simplifies to $n^k/k!$. This makes sense because, if $n/(n-k)$ is essentially 1, then the product of the k largest factors in $n!$ is essentially n^k.

1.8 Combinatorial Methods

Commentary

This section ends with an extended example called "The Tennis Tournament". This is an application of combinatorics that uses a slightly subtle line of reasoning.

Solutions to Exercises

1. We have to assign 10 houses to one pollster, and the other pollster will get to canvas the other 10 houses. Hence, the number of assignments is the number of combinations of 20 items taken 10 at a time,

$$\binom{20}{10} = 184{,}756.$$

2. The ratio of $\binom{93}{30}$ to $\binom{93}{31}$ is $31/63 < 1$, so $\binom{93}{31}$ is larger.

3. Since $93 = 63 + 30$, the two numbers are the same.

4. Let the sample space consist of all subsets (not ordered tuples) of the 24 bulbs in the box. There are $\binom{24}{4} = 10626$ such subsets. There is only one subset that has all four defectives, so the probability we want is $1/10626$.

5. The number is $\dfrac{4251!}{(97!4154!)} = \dbinom{4251}{97}$, an integer.

6. There are $\binom{n}{2}$ possible pairs of seats that A and B can occupy. Of these pairs, $n-1$ pairs comprise two adjacent seats. Therefore, the probability is $\dfrac{n-1}{\binom{n}{2}} = \dfrac{2}{n}$.

7. There are $\binom{n}{k}$ possible sets of k seats to be occupied, and they are all equally likely. There are $n-k+1$ sets of k adjacent seats, so the probability we want is

$$\frac{n-k+1}{\binom{n}{k}} = \frac{(n-k+1)!k!}{n!}.$$

8. There are $\binom{n}{k}$ possible sets of k seats to be occupied, and they are all equally likely. Because the circle has no start or end, there are n sets of k adjacent seats, so the probability we want is

$$\frac{n}{\binom{n}{k}} = \frac{(n-k)!k!}{(n-1)!}.$$

9. This problem is slightly tricky. The total number of ways of choosing the n seats that will be occupied by the n people is $\binom{2n}{n}$. Offhand, it would seem that there are only two ways of choosing these seats so that no two adjacent seats are occupied, namely:

X0X0 . . . 0 and 0X0X . . . 0X

Upon further consideration, however, $n-1$ more ways can be found, namely:

X00X0X . . . 0X, X0X00X0X . . . 0X, etc.

Therefore, the total number of ways of choosing the seats so that no two adjacent seats are occupied is $n+1$. The probability is $(n+1)/\binom{2n}{n}$.

10. We shall let the sample space consist of all subsets (unordered) of 10 out of the 24 light bulbs in the box. There are $\binom{24}{10}$ such subsets. The number of subsets that contain the two defective bulbs is the number of subsets of size 8 out of the other 22 bulbs, $\binom{22}{8}$, so the probability we want is

$$\frac{\binom{22}{8}}{\binom{24}{10}} = \frac{10 \times 9}{24 \times 23} = 0.1630.$$

11. This exercise is similar to Exercise 10. Let the sample space consist of all subsets (unordered) of 12 out of the 100 people in the group. There are $\binom{100}{12}$ such subsets. The number of subsets that contain A and B is the number of subsets of size 10 out of the other 98 people, $\binom{98}{10}$, so the probability we want is

$$\frac{\binom{98}{10}}{\binom{100}{12}} = \frac{12 \times 11}{100 \times 99} = 0.01333.$$

12. There are $\binom{35}{10}$ ways of dividing the group into the two teams. As in Exercise 11, the number of ways of choosing the 10 players for the first team so as to include both A and B is $\binom{33}{8}$. The number of ways of choosing the 10 players for this team so as not to include either A or B (A and B will then be together on the other team) is $\binom{33}{10}$. The probability we want is then

$$\frac{\binom{33}{8} + \binom{33}{10}}{\binom{35}{10}} = \frac{10 \times 9 + 25 \times 24}{35 \times 34} = 0.5798.$$

13. This exercise is similar to Exercise 12. Here, we want four designated bulbs to be in the same group. The probability is

$$\frac{\binom{20}{6} + \binom{20}{10}}{\binom{24}{10}} = 0.1140.$$

14.

$$\begin{aligned}\binom{n}{k}+\binom{n}{k-1} &= \frac{n!}{k!(n-k)!}+\frac{n!}{(k-1)!(n-k+1)!}\\ &= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{k}+\frac{1}{n-k+1}\right)\\ &= \frac{n!}{(k-1)!(n-k)!}\cdot\frac{n+1}{k(n-k+1)}\\ &= \frac{(n+1)!}{k!(n-k+1)!}=\binom{n+1}{k}.\end{aligned}$$

15. (a) If we express 2^n as $(1+1)^n$ and expand $(1+1)^n$ by the binomial theorem, we obtain the desired result.

    (b) If we express 0 as $(1-1)^n$ and expand $(1-1)^n$ by the binomial theorem, we obtain the desired result.

16. (a) It is easier to calculate first the probability that the committee will not contain either of the two senators from the designated state. This probability is $\binom{98}{8}\Big/\binom{100}{8}$. Thus, the final answer is

$$1-\frac{\binom{98}{8}}{\binom{100}{8}}\approx 1-.08546=0.1543.$$

    (b) There are $\binom{100}{50}$ combinations that might be chosen. If the group is to contain one senator from each state, then there are two possible choices for each of the fifty states. Hence, the number of possible combinations containing one senator from each state is 2^{50}.

17. Call the four players A, B, C, and D. The number of ways of choosing the positions in the deck that will be occupied by the four aces is $\binom{52}{4}$. Since player A will receive 13 cards, the number of ways of choosing the positions in the deck for the four aces so that all of them will be received by player A is $\binom{13}{4}$. Similarly, since player B will receive 13 other cards, the number of ways of choosing the positions for the four aces so that all of them will be received by player B is $\binom{13}{4}$. A similar result is true for each of the other players. Therefore, the total number of ways of choosing the positions in the deck for the four aces so that all of them will be received by the same player is $4\binom{13}{4}$. Thus, the final probability is $4\binom{13}{4}\Big/\binom{52}{4}$.

18. There are $\binom{100}{10}$ ways of choosing ten mathematics students. There are $\binom{20}{2}$ ways of choosing two

students from a given class of 20 students. Therefore, there are $\binom{20}{2}^5$ ways of choosing two students from each of the five classes. So, the final answer is $\binom{20}{2}^5 / \binom{100}{10} \approx 0.0143$.

19. From the description of what counts as a collection of customer choices, we see that each collection consists of a tuple $(m_1, \ldots, m_n)$, where m_i is the number of customers who choose item i for $i = 1, \ldots, n$. Each m_i must be between 0 and k and $m_1 + \cdots + m_n = k$. Each such tuple is equivalent to a sequence of $n + k - 1$ 0's and 1's as follows. The first m_1 terms are 0 followed by a 1. The next m_2 terms are 0 followed by a 1, and so on up to m_{n-1} 0's followed by a 1 and finally m_n 0's. Since $m_1 + \cdots + m_n = k$ and since we are putting exactly $n - 1$ 1's into the sequence, each such sequence has exactly $n + k - 1$ terms. Also, it is clear that each such sequence corresponds to exactly one tuple of customer choices. The numbers of 0's between successive 1's give the numbers of customers who choose that item, and the 1's indicate where we switch from one item to the next. So, the number of combinations of choices is the number of such sequences: $\binom{n+k-1}{k}$.

20. We shall use induction. For $n = 1$, we must prove that

$$x + y = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0.$$

Since the right side of this equation is $x + y$, the theorem is true for $n = 1$. Now assume that the theorem is true for each $n = 1, \ldots, n_0$ for $n_0 \geq 1$. For $n = n_0 + 1$, the theorem says

$$(x+y)^{n_0+1} = \sum_{k=0}^{n_0+1} \binom{n_0+1}{k} x^k y^{n_0+1-k}. \tag{S.1.2}$$

Since we have assumed that the theorem is true for $n = n_0$, we know that

$$(x+y)^{n_0} = \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0-k}. \tag{S.1.3}$$

We shall multiply both sides of (S.1.3) by $x + y$. We then need to prove that $x + y$ times the right side of (S.1.3) equals the right side of (S.1.2).

$$\begin{aligned}
(x+y)(x+y)^{n_0} &= (x+y) \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0-k} \\
&= \sum_{k=0}^{n_0} \binom{n_0}{k} x^{k+1} y^{n_0-k} + \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0+1-k} \\
&= \sum_{k=1}^{n_0+1} \binom{n_0}{k-1} x^k y^{n_0+1-k} + \sum_{k=0}^{n_0} \binom{n_0}{k} x^k y^{n_0+1-k} \\
&= y^{n_0+1} + \sum_{k=1}^{n_0} \left[\binom{n_0}{k-1} + \binom{n_0}{k} \right] x^k y^{n_0+1-k} + x^{n_0+1}.
\end{aligned}$$

Now, apply the result in Exercise 14 to conclude that

$$\binom{n_0}{k-1} + \binom{n_0}{k} = \binom{n_0+1}{k}.$$