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Chapter 1

- 1.1. A finite element is a small body or unit interconnected to other units to model a larger structure or system.
- 1.2. Discretization means dividing the body (system) into an equivalent system of finite elements with associated nodes and elements.
- 1.3. The modern development of the finite element method began in 1941 with the work of Hrennikoff in the field of structural engineering.
- 1.4. The direct stiffness method was introduced in 1941 by Hrennikoff. However, it was not commonly known as the direct stiffness method until 1956.
- 1.5. A matrix is a rectangular array of quantities arranged in rows and columns that is often used to aid in expressing and solving a system of algebraic equations.
- 1.6. As computer developed it made possible to solve thousands of equations in a matter of minutes.
- 1.7. The following are the general steps of the finite element method.
 - Step 1
Divide the body into an equivalent system of finite elements with associated nodes and choose the most appropriate element type.
 - Step 2
Choose a displacement function within each element.
 - Step 3
Relate the stresses to the strains through the stress/strain law—generally called the constitutive law.
 - Step 4
Derive the element stiffness matrix and equations. Use the direct equilibrium method, a work or energy method, or a method of weighted residuals to relate the nodal forces to nodal displacements.
 - Step 5
Assemble the element equations to obtain the global or total equations and introduce boundary conditions.
 - Step 6
Solve for the unknown degrees of freedom (or generalized displacements).
 - Step 7
Solve for the element strains and stresses.
 - Step 8
Interpret and analyze the results for use in the design/analysis process.
- 1.8. The displacement method assumes displacements of the nodes as the unknowns of the problem. The problem is formulated such that a set of simultaneous equations is solved for nodal displacements.
- 1.9. Four common types of elements are: simple line elements, simple two-dimensional elements, simple three-dimensional elements, and simple axisymmetric elements.
- 1.10. Three common methods used to derive the element stiffness matrix and equations are
 - (1) direct equilibrium method
 - (2) work or energy methods
 - (3) methods of weighted residuals
- 1.11. The term ‘degrees of freedom’ refers to rotations and displacements that are associated with each node.

1.12. Five typical areas where the finite element is applied are as follows.

- (1) Structural/stress analysis
- (2) Heat transfer analysis
- (3) Fluid flow analysis
- (4) Electric or magnetic potential distribution analysis
- (5) Biomechanical engineering

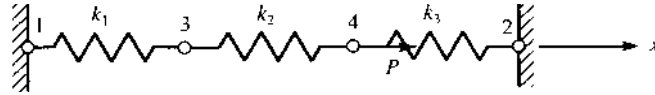
1.13. Five advantages of the finite element method are the ability to

- (1) Model irregularly shaped bodies quite easily
- (2) Handle general load conditions without difficulty
- (3) Model bodies composed of several different materials because element equations are evaluated individually
- (4) Handle unlimited numbers and kinds of boundary conditions
- (5) Vary the size of the elements to make it possible to use small elements where necessary

Chapter 2

2.1

(a)



$$[k^{(1)}] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 \\ -k_1 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[k^{(2)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_2 & -k_2 \\ 0 & 0 & -k_2 & k_2 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k_3 & 0 & -k_3 \\ 0 & 0 & 0 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$$

$$[K] = [k^{(1)}] + [k^{(2)}] + [k^{(3)}]$$

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix}$$

(b) Nodes 1 and 2 are fixed so $u_1 = 0$ and $u_2 = 0$ and $[K]$ becomes

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$\{F\} = [K] \{d\}$$

$$\begin{Bmatrix} F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$\{F\} = [K] \{d\} \Rightarrow [K^{-1}] \{F\} = [K^{-1}] [K] \{d\}$$

$$\Rightarrow [K^{-1}] \{F\} = \{d\}$$

Using the adjoint method to find $[K^{-1}]$

$$C_{11} = k_2 + k_3 \qquad C_{21} = (-1)^3 (-k_2)$$

$$C_{12} = (-1)^{1+2} (-k_2) = k_2 \qquad C_{22} = k_1 + k_2$$

$$[C] = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \text{ and } C^T = \begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}$$

$$\det [K] = |[K]| = (k_1 + k_2)(k_2 + k_3) - (-k_2)(-k_2)$$

$$\Rightarrow |[K]| = (k_1 + k_2)(k_2 + k_3) - k_2^2$$

$$[K^{-1}] = \frac{[C^T]}{\det K}$$

$$[K^{-1}] = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{(k_1 + k_2)(k_2 + k_3) - k_2^2} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} = \frac{\begin{bmatrix} k_2 + k_3 & k_2 \\ k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ P \end{Bmatrix}}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow u_3 = \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow u_4 = \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

(c) In order to find the reaction forces we go back to the global matrix $F = [K] \{d\}$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

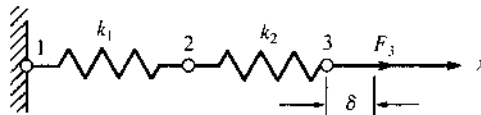
$$F_{1x} = -k_1 u_3 = -k_1 \frac{k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{1x} = \frac{-k_1 k_2 P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$F_{2x} = -k_3 u_4 = -k_3 \frac{(k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

$$\Rightarrow F_{2x} = \frac{-k_3 (k_1 + k_2) P}{k_1 k_2 + k_1 k_3 + k_2 k_3}$$

2.2



$$k_1 = k_2 = k_3 = 1000 \frac{\text{lb}}{\text{in.}}$$

$$[k^{(1)}] = \begin{matrix} & \begin{matrix} (1) & (2) \end{matrix} \\ \begin{matrix} (1) \\ (2) \end{matrix} & \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}; \quad [k^{(2)}] = \begin{matrix} & \begin{matrix} (2) & (3) \end{matrix} \\ \begin{matrix} (2) \\ (3) \end{matrix} & \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

By the method of superposition the global stiffness matrix is constructed.

$$\begin{matrix}
 (1) & (2) & (3) \\
 [K] = \begin{bmatrix} k & -k & 0 \\ -k & k+k & -k \\ 0 & -k & k \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \Rightarrow [K] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}
 \end{matrix}$$

Node 1 is fixed $\Rightarrow u_1 = 0$ and $u_3 = \delta$

$$\{F\} = [K] \{d\}$$

$$\begin{matrix}
 \begin{matrix} F_{1x}=? \\ F_{2x}=0 \\ F_{3x}=? \end{matrix} \\
 \end{matrix} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{matrix} u_1=0 \\ u_2=? \\ u_3=\delta \end{matrix}$$

$$\Rightarrow \begin{Bmatrix} 0 \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2 \\ \delta \end{Bmatrix} \Rightarrow \begin{cases} 0 = 2k u_2 - k \delta \\ F_{3x} = -k u_2 + k \delta \end{cases}$$

$$\Rightarrow u_2 = \frac{k\delta}{2k} = \frac{\delta}{2} = \frac{1 \text{ in.}}{2} \Rightarrow u_2 = 0.5''$$

$$F_{3x} = -k(0.5'') + k(1'')$$

$$F_{3x} = \left(-1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') + \left(1000 \frac{\text{lb}}{\text{in.}}\right)(1'')$$

$$F_{3x} = 500 \text{ lbs}$$

Internal forces

Element (1)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1=0 \\ u_2=0.5'' \end{Bmatrix}$$

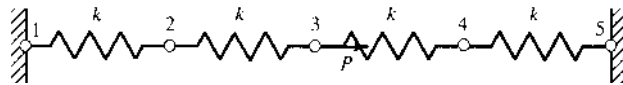
$$\Rightarrow f_{1x}^{(1)} = \left(-1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') \Rightarrow f_{1x}^{(1)} = -500 \text{ lb}$$

$$f_{2x}^{(1)} = \left(1000 \frac{\text{lb}}{\text{in.}}\right)(0.5'') \Rightarrow f_{2x}^{(1)} = 500 \text{ lb}$$

Element (2)

$$\begin{Bmatrix} f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_2=0.5'' \\ u_3=1'' \end{Bmatrix} \Rightarrow \begin{cases} f_{2x}^{(2)} = -500 \text{ lb} \\ f_{3x}^{(2)} = 500 \text{ lb} \end{cases}$$

2.3



$$(a) [k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

By the method of superposition we construct the global $[K]$ and knowing $\{F\} = [K] \{d\}$ we have

$$\begin{matrix}
 \begin{matrix} F_{1x}=? \\ F_{2x}=0 \\ F_{3x}=P \\ F_{4x}=0 \\ F_{5x}=? \end{matrix} \\
 \end{matrix} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{matrix} u_1=0 \\ u_2 \\ u_3 \\ u_4 \\ u_5=0 \end{matrix}$$

$$(b) \begin{cases} 0 \\ P \\ 0 \end{cases} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{cases} u_2 \\ u_3 \\ u_4 \end{cases} \Rightarrow \begin{cases} 0 = 2ku_2 - ku_3 \\ P = -ku_2 + 2ku_3 - ku_4 \\ 0 = -ku_3 + 2ku_4 \end{cases}$$

$$\Rightarrow u_2 = \frac{u_3}{2}; u_4 = \frac{u_3}{2}$$

Substituting in the equation in the middle

$$P = -k u_2 + 2k u_3 - k u_4$$

$$\Rightarrow P = -k \left(\frac{u_3}{2} \right) + 2k u_3 - k \left(\frac{u_3}{2} \right)$$

$$\Rightarrow P = k u_3$$

$$\Rightarrow u_3 = \frac{P}{k}$$

$$u_2 = \frac{P}{2k}; u_4 = \frac{P}{2k}$$

(c) In order to find the reactions at the fixed nodes 1 and 5 we go back to the global equation $\{F\} = [K] \{d\}$

$$F_{1x} = -k u_2 = -k \frac{P}{2k} \Rightarrow F_{1x} = -\frac{P}{2}$$

$$F_{5x} = -k u_4 = -k \frac{P}{2k} \Rightarrow F_{5x} = -\frac{P}{2}$$

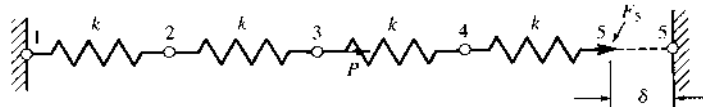
Check

$$\Sigma F_x = 0 \Rightarrow F_{1x} + F_{5x} + P = 0$$

$$\Rightarrow -\frac{P}{2} + \left(-\frac{P}{2} \right) + P = 0$$

$$\Rightarrow 0 = 0$$

2.4



$$(a) [k^{(1)}] = [k^{(2)}] = [k^{(3)}] = [k^{(4)}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

By the method of superposition the global $[K]$ is constructed.

Also $\{F\} = [K] \{d\}$ and $u_1 = 0$ and $u_5 = \delta$

$$\begin{cases} F_{1x} = ? \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 0 \\ F_{5x} = ? \end{cases} = \begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix} \begin{cases} u_1 = 0 \\ u_2 = ? \\ u_3 = ? \\ u_4 = ? \\ u_5 = \delta \end{cases}$$

$$(b) \quad 0 = 2k u_2 - k u_3 \quad (1)$$

$$0 = -k u_2 + 2k u_3 - k u_4 \quad (2)$$

$$0 = -k u_3 + 2k u_4 - k \delta \quad (3)$$

From (2)

$$u_3 = 2 u_2$$

From (3)

$$u_4 = \frac{\delta + 2 u_2}{2}$$

Substituting in Equation (2)

$$\Rightarrow -k(u_2) + 2k(2u_2) - k\left(\frac{\delta + 2\delta_{2x}}{2}\right)$$

$$\Rightarrow -u_2 + 4u_2 - u_2 - \frac{\delta}{2} = 0 \Rightarrow u_2 = \frac{\delta}{4}$$

$$\Rightarrow u_3 = 2 \frac{\delta}{4} \Rightarrow u_3 = \frac{\delta}{2}$$

$$\Rightarrow u_4 = \frac{\delta + 2\left(\frac{\delta}{4}\right)}{2} \Rightarrow u_4 = \frac{3\delta}{4}$$

(c) Going back to the global equation

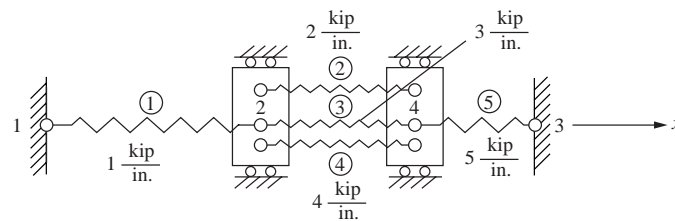
$$\{F\} = [K] \{d\}$$

$$F_{1x} = -k u_2 = -k \frac{\delta}{4} \Rightarrow F_{1x} = -\frac{k \delta}{4}$$

$$F_{5x} = -k u_4 + k \delta = -k \left(\frac{3\delta}{4}\right) + k \delta$$

$$\Rightarrow F_{5x} = \frac{k \delta}{4}$$

2.5



$$[k^{(1)}] = \begin{bmatrix} d_1 & d_2 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad [k^{(2)}] = \begin{bmatrix} d_2 & d_4 \\ 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$[k^{(3)}] = \begin{bmatrix} d_2 & d_4 \\ 3 & -3 \\ -3 & 3 \end{bmatrix}; \quad [k^{(4)}] = \begin{bmatrix} d_2 & d_4 \\ 4 & -4 \\ -4 & 4 \end{bmatrix}$$

$$[k^{(5)}] = \begin{bmatrix} d_4 & d_3 \\ 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Assembling global $[K]$ using direct stiffness method

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2+3+4 & 0 & -2-3-4 \\ 0 & 0 & 5 & -5 \\ 0 & -2-3-4 & -5 & 2+3+4+5 \end{bmatrix}$$

Simplifying

$$[K] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \frac{\text{kip}}{\text{in.}}$$

2.6 Now apply + 2 kip at node 2 in spring assemblage of P 2.5.

$$\therefore F_{2x} = 2 \text{ kip}$$

$$[K]\{d\} = \{F\}$$

$[K]$ from P 2.5

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 10 & 0 & -9 \\ 0 & 0 & 5 & -5 \\ 0 & -9 & -5 & 14 \end{bmatrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 2 \\ F_3 \\ 0 \end{Bmatrix} \quad (\text{A})$$

where $u_1 = 0, u_3 = 0$ as nodes 1 and 3 are fixed.

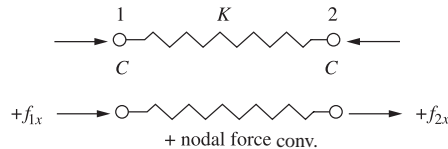
Using Equations (1) and (3) of (A)

$$\begin{bmatrix} 10 & -9 \\ -9 & 14 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

Solving

$$u_2 = 0.475 \text{ in.}, \quad u_4 = 0.305 \text{ in.}$$

2.7



$$f_{1x} = C, \quad f_{2x} = -C$$

$$f = -k\delta = -k(u_2 - u_1)$$

$$\therefore f_{1x} = -k(u_2 - u_1)$$

$$f_{2x} = -(-k)(u_2 - u_1)$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore [K] = \begin{Bmatrix} k & -k \\ -k & k \end{Bmatrix} \text{ same as for tensile element}$$