Solutions to Problems in Goldstein,
Classical Mechanics,

Chapter 1

Problem 1.1

A nucleus, originally at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c, and at right angles to the direction of the electron a neutrino with momentum 1.00 MeV/c. (The MeV (million electron volt) is a unit of energy, used in modern physics, equal to 1.60 x 10^{-6} erg. Correspondingly, MeV/c is a unit of linear momentum equal to 5.34 x 10^{-17} gm-cm/sec.) In what direction does the nucleus recoil? What is its momentum in MeV/c? If the mass of the residual nucleus is 3.90 x 10^{-22} gm, what is its kinetic energy, in electron volts?

Place the nucleus at the origin, and suppose the electron is emitted in the positive y direction, and the neutrino in the positive x direction. Then the resultant of the electron and neutrino momenta has magnitude

$$|\mathbf{p}_{\mathbf{e}+\nu}| = \sqrt{(1.73)^2 + 1^2} = 2 \text{ MeV/c},$$

and its direction makes an angle

$$\theta = \tan^{-1}\frac{1.73}{1} = 60^\circ$$

with the x axis. The nucleus must acquire a momentum of equal magnitude and directed in the opposite direction. The kinetic energy of the nucleus is

$$T = \frac{p^2}{2m} = \frac{4 \text{ MeV}^2 \text{ c}^{-2}}{2 \cdot 3.9 \cdot 10^{-22} \text{ gm}} \cdot \frac{1.78 \cdot 10^{-27} \text{ gm}}{1 \text{ MeV c}^{-2}} = 9.1 \text{ ev}$$

This is much smaller than the nucleus rest energy of several hundred GeV, so the non-relativistic approximation is justified.

Problem 1.2

The *escape velocity* of a particle on the earth is the minimum velocity required at the surface of the earth in order that the particle can escape from the earth's gravitational field. Neglecting the resistance of the atmosphere, the system is conservative. From the conservation theorem for potential plus kinetic energy show that the escape velocity for the earth, ignoring the presence of the moon, is 6.95 mi/sec.

If the particle starts at the earth's surface with the escape velocity, it will just manage to break free of the earth's field and have nothing left. Thus after it has escaped the earth's field it will have no kinetic energy left, and also no potential energy since it's out of the earth's field, so its total energy will be zero. Since the particle's total energy must be constant, it must also have zero total energy at the surface of the earth. This means that the kinetic energy it has at the surface of the earth must exactly cancel the gravitational potential energy it has there:

$$\frac{1}{2}mv_e^2 - G\frac{mM_R}{R_R} = 0$$

so

$$\begin{aligned} v = \sqrt{\left(\frac{2GM_R}{R_R}\right)} &= \left(\frac{2\cdot(6.67\cdot 10^{11}\ \mathrm{m}^3\ \mathrm{kg}^{-3}\ \mathrm{s}^{-2})\cdot(5.98\cdot 10^{24}\ \mathrm{kg})}{6.38\cdot 10^{6}\ \mathrm{m}}\right)^{1/2} \\ &= 11.2\ \mathrm{km/s}\cdot\frac{1\ \mathrm{m}}{1.61\ \mathrm{km}} = 6.95\ \mathrm{mi/s}. \end{aligned}$$

Problem 1.3

Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail. Since these gases arise from the reaction of the fuels carried in the rocket the mass of the rocket is not constant, but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field, neglecting atmospheric resistance, is

$$m\frac{dv}{dt} = -v'\frac{dm}{dt} - mg,$$

where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Integrate this equation to obtain v as a function of m, assuming a constant time rate of loss of mass. Show, for a rocket starting initially from rest, with v' equal to 6800 ft/sec and a mass loss per second equal to 1/60th of the initial mass, that in order to reach the escape velocity the ratio of the weight of the fuel to the weight of the empty rocket must be almost 300!

Suppose that, at time t, the rocket has mass $m(t)$ and velocity $v(t)$. The total external force on the rocket is then $F = gm(t)$, with $g = 32.1$ ft/s^2, pointed downwards, so that the total change in momentum between t and $t + dt$ is

$$F dt = -gm(t)dt. \tag{1}$$

At time t, the rocket has momentum

$$p(t) = m(t)v(t). \tag{2}$$

On the other hand, during the time interval dt the rocket releases a mass Δm of gas at a velocity v' with respect to the rocket. In so doing, the rocket's velocity increases by an amount dv. The total momentum at time $t + dt$ is the sum of the momenta of the rocket and gas:

$$p(t + dt) = p_r + p_g = [m(t) - \Delta m][v(t) + dv] + \Delta m[v(t) + v'] \tag{3}$$

Subtracting (2) from (3) and equating the difference with (1), we have (to first order in differential quantities)

$$-gm(t)dt = m(t)dv + v'\Delta m$$

or

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\frac{\Delta m}{dt}$$

which we may write as

$$\frac{dv}{dt} = -g - \frac{v'}{m(t)}\gamma \tag{4}$$

where

$$\gamma = \frac{\Delta m}{dt} = \frac{1}{60} m_0 s^{-1}.$$

This is a differential equation for the function $v(t)$ giving the velocity of the rocket as a function of time. We would now like to recast this as a differential equation for the function $v(m)$ giving the rocket's velocity as a function of its mass. To do this, we first observe that since the rocket is *releasing* the mass Δm every dt seconds, the time derivative of the rocket's mass is

$$\frac{dm}{dt} = -\frac{\Delta m}{dt} = -\gamma.$$

We then have

$$\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt} = -\gamma\frac{dv}{dm}.$$

Substituting into (4), we obtain

$$-\gamma\frac{dv}{dm} = -g - \frac{v'}{m}\gamma$$

or

$$dv = \frac{g}{\gamma} dm + v' \frac{dm}{m}.$$

Integrating, with the condition that $v(m_0) = 0$,

$$v(m) = \frac{g}{\gamma}(m - m_0) + v' \ln\left(\frac{m}{m_0}\right).$$

Now, γ=(1/60)m_0 s^{-1}, while v' =-6800 ft/s. Then

$$v(m) = 1930 \text{ ft/s} \cdot \left(\frac{m}{m_0} - 1\right) + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right)$$

For $m_0 \gg m$ we can neglect the first term in the parentheses of the first term, giving

$$v(m) = -1930 \text{ ft/s} + 6800 \text{ ft/s} \cdot \ln\left(\frac{m_0}{m}\right).$$

The escape velocity is $v = 6.95$ mi/s $= 36.7 \cdot 10^3$ ft/s. Plugging this into the equation above and working backwards, we find that escape velocity is achieved when m_0/m=293.

Thanks to Brian Hart for pointing out an inconsistency in my original choice of notation for this problem.

Problem 1.4

Show that for a single particle with constant mass the equation of motion implies the following differential equation for the kinetic energy:

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v},$$

while if the mass varies with time the corresponding equation is

$$\frac{d(mT)}{dt} = \mathbf{F} \cdot \mathbf{p}.$$

We have

$$\mathbf{F} = \dot{\mathbf{p}} \tag{5}$$

If m is constant,

$$\mathbf{F} = m\dot{\mathbf{v}}$$

Dotting $\mathbf{v}$ into both sides,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{v} = m\mathbf{v} \cdot \dot{\mathbf{v}} &= \frac{1}{2} m \frac{d}{dt} |\mathbf{v}|^2 \\ &= \frac{dT}{dt} \end{aligned} \tag{6}$$

On the other hand, if m is not constant, instead of $\mathbf{v}$ we dot $\mathbf{p}$ into (5):

$$\begin{aligned} \mathbf{F} \cdot \mathbf{p} &= \mathbf{p} \cdot \dot{\mathbf{p}} \\ &= m\mathbf{v} \cdot \frac{d(m\mathbf{v})}{dt} \\ &= m\mathbf{v} \cdot \left(\mathbf{v} \frac{dm}{dt} + m \frac{d\mathbf{v}}{dt} \right) \\ &= \frac{1}{2} v^2 \frac{d}{dt} m^2 + \frac{1}{2} m^2 \frac{d}{dt} (v^2) \\ &= \frac{1}{2} \frac{d}{dt} (m^2 v^2) = \frac{d(mT)}{dt}. \end{aligned}$$